Lecture 7, Solar Radiation, Part 3, Earth-Sun Geometry

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Topics to be Covered

1. Lambert's cosine law
2. Earth-Sun geometry
   a. Earth declination, elevation, zenith, hour and azimuth angles
   b. seasons
   c. sunrise and sunset times
3. View factors and Shadows
4. Radiation on inclined surfaces, mountain slopes, etc

L7.1 Introduction

We need to assess the flux density of solar radiation on flat and inclined surfaces for a variety of biometeorological problems. This information is used to compute the energy balance, evaporation and photosynthesis rates and stomatal conductance of inclined leaves, and plants. In this lecture we will cover issues relating to cosine laws between the incident and received rays and learn to compute angles between the sun and the normal to points on earth.

L7.2 Lambert's Cosine Law

The flux density radiation on the horizontal is a function of the angle between the surface normal and the direction of the ray is defined by Lambert’s Cosine Law:

\[ R(i) = R_\perp \cos(\gamma) = R_\perp \sin(\beta) \]
Figure 1 Definition of zenith and elevation angles and the projection of area normal to incident rays on a flat surface.

The ratio between the flux density of radiation on the horizontal surface to the flux density normal to the incoming radiation is related to the ratios of the areas of the beam cross section ($A_{\text{beam}}$) and the projected horizontal area ($A_{\text{horizontal}}$).

$$R(i) = R_i \frac{A_{\text{beam}}}{A_{\text{horizontal}}}$$

Trigonometrically, we can examine 2 complementary triangles and consider the projection of side B onto side C.
To calculate the flux density of solar radiation on the horizontal, we want to project the flux density of radiation associated with the vector $\mathbf{C}$ to that normal to the surface, vector $\mathbf{A}$.

For the lower triangle and angle $\beta$ we arrive at:

$$\frac{R(i)}{R_\perp} = \frac{A}{C} = \sin(\beta)$$

$$R(i) = R_\perp \sin(\beta) = R_\perp \frac{A}{C}$$

And for the upper triangle and angle $\gamma$ we arrive at:

$$\cos(\gamma) = \frac{A}{C}$$

$$R(i)\cos(\gamma) = R_\perp \frac{A}{C}$$

The lower the elevation angle, the more the unit area of beam is spread out on the ground, so its flux density on an unit area basis is reduced. As we can see in Figure 2, $\cos(\gamma)$ is one when $\gamma$ is zero and the sun is directly overhead; $\cos(\gamma)$ is zero when $\gamma$ is 90 degrees and the sun is aligned with the horizon.
L7.3 Sun-Earth Geometry

The Earth rotates on its axis and it revolves around the Sun in an elliptical orbit. The area swept by the Earth-Sun radius is constant, as predicted by Kepler’s Law of Planetary Motion. To perform calculations associated with the flux density of solar radiation on flat and inclined surfaces and its penetration into plant canopy, we must be able to evaluate the angle of the sun from any position on Earth at any time of the day. Such a calculation involves computing the angle between a vector normal to a specified point on Earth and a projection from the Sun. This computation is complicated by the fact that the Earth rotates on its polar axis once a day and the Earth revolves around the Sun once a year.

To compute the sun's elevation with respect to the horizon (or its complement, the angle with to the zenith) and its azimuth angle we apply concepts derived from three-dimensional geometry (Campbell, Norman, 1998; Monteith, Unsworth, 1990). The important coordinates to know include:

1. **time of day**
2. **longitude**
3. **latitude** ($\lambda$)
4. **Earth’s declination angle** ($\delta$).

One way to think about the problem is to try and solve for the angle between two vectors. One is the normal to the Earth at a given position and the other is the vector between the
point on Earth that is normal to the view ($\vec{n}$) and the vector that points to the Sun ($\vec{S}$). From Vector analysis we know that the cosine of the vectors is related to their dot product

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \gamma$$

A schematic of the situation we are interested shows three vectors, $\vec{n}$, $\vec{S}$, and $\vec{S} - \vec{n}$, which represent the vector equal to the radius of the Earth and directed toward the surface zenith, the vector from the center of the Earth to the Sun, and the resultant vector. It is the angle between the resultant vector and the vector pointing towards the zenith that gives us the angle we desire.

After much algebraic and trigonometric expansion and substitution we arrive at a new equation of the solar zenith or elevation angles. The cosine of the sun’s angle with respect to the zenith ($\gamma$) or the sine of the solar elevation angle ($\beta$) with respect to the horizontal, can be computed as:

$$\beta = \sin(\lambda) \sin(\delta) + \cos(\lambda) \cos(\delta) \cos(h) = \cos(\gamma)$$

$\lambda$ is latitude, $h$ is the hour angle and $\delta$ is declination angle.

We discuss the components of computing Earth-Sun angles in more detail.

Time of day

The Earth is a rotating sphere. The Earth rotates around its north-south axis once every 23 hours, 56 minutes and 4 second.
It can also be stated that the Earth rotates on its axis $2\pi$ radians in one day. Over the course of an hour it rotates $\pi/12$ radians or 15 degrees. In this framework, the hour angle, $h$, is defined as:

$$h = \frac{2\pi t}{24} = \frac{\pi t}{12}$$

The following table produces values of $h$ and $\cos(h)$ for several reference values:

<table>
<thead>
<tr>
<th>Solar time (hour)</th>
<th>$h$ (radians)</th>
<th>$\cos(h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>$0.5\pi$</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>$\pi$</td>
<td>-1</td>
</tr>
<tr>
<td>18</td>
<td>$1.5\pi$</td>
<td>0</td>
</tr>
</tbody>
</table>

In the northern hemisphere, the cosine of $\pi$ equalling minus one corresponds with the sun being south of an observer at solar noon.

If we are to apply this relation using clock time, then we define $h$ as the fraction of $2\pi$ that the earth has turned after local solar noon:

$$h = \frac{\pi}{12} (t - t_0) \text{ (radians)}$$

$$h = \frac{15\pi}{180} (t - t_0) \text{ (degrees)}$$

In this instance, $t$ is local time and $t_0$ is solar time.

$$t_0 = 12 - l_c - e_t$$

$l_c$ is the longitudinal correction. Its value is 4 minutes for each degree of longitude east of the standard meridian and is -4 minutes for each degree of longitude west of the standard meridian. Standard meridians possess 15 degree increments from the prime meridian (Campbell, Norman, 1998).

The longitude of the Berkeley campus is: 122° W 15’ 47”; latitude is 37° N 52’ 24”. It is about 2.25° west of the standard meridian (120° W), so solar noon occurs about 9 minutes after noon Pacific Standard Time.
The variable, $e_t$, represents the equation of time. It arises because of the eccentricity of the Earth’s orbit around the sun and the obliquity (tilt) of the ecliptic (the great circle that represents the Earth’s path around the sun) (Campbell, Norman, 1998). The eccentricity of the earth orbit around the sun forces the angular rotation rate to be variable; this allows equal areas to be swept with time during the Earth’s orbit. The Equation of Time (units of hours) is:

$$e_t = \left[ \frac{-104.7\sin(f) + 596.2\sin(2f) + 4.3\sin(3f) - 12.7\sin(4f) - 429.3\cos(f) - 2.0\cos(2f) + 19.3\cos(3f)}{3600} \right]$$

$$f = \frac{\pi}{180} \left[ 279.5 + 0.9856d \right] \text{ (radians)}$$

$d$ is day number (1 on Jan. 1 and 365 on Dec. 31). Here $e_t$ and $f$ have units of radians.

The contribution from variation in the orbital speed of the Earth to the computation of the Sun’s zenith angle is not trivial. On February 9, the increment due to the equation of time is – 14.2 minutes and on November 6 the contribution is 16.3 minutes.

![Figure 4 Computations based on the equation of time (adapted from Campbell and Norman, 1998)](image-url)

Sometimes these equations are expressed in terms of Universal Time. In this case, the universal time (Greenwich) is the sum of the true solar time plus and equation for time
and the longitudinal time (15 degrees per hour West of Greenwich and –15 degrees per hour East of Greenwich).

**Longitude**

Mariners, navigators and geographers have divided the sphere of the Earth in the north-south plane into zones of latitude. They correspond with the degrees of a circle (0 to 360 degrees or 0 to $2\pi$ radians). The reference for zero is the prime meridian, near London, at Greenwich, England.

**Latitude**

From an imaginary point at the center of the Earth’s sphere angular bands of latitude are defined. At the center is the sphere is the Equator, defined as zero degrees. Bands of latitude range from zero to 90 degrees for the northern and southern hemispheres.

**Earth’s Declination Angle**

The Earth's axis of rotation is tilted 66.5 degrees with respect to its orbital plane around the sun and its axis of rotation is inclined 23.5 degrees from the perpendicular, with respect to this plane. The tilt of the Earth affects the angle between the sun beam and the normal over a surface. The sun angle affects the flux density of solar radiation incident on a surface.

The declination angle corresponds with the angle between the sun's rays at solar noon and the equator. The Earth’s declination angle has limits that correspond with the seasons:

Summer solstice: + 23.45 degrees (June 22)

![Figure 5 Geometrical configuration between the Sun and the Earth during the Summer Solstice](image)
During the summer solstice the sun is directly overhead (90° degrees) at the Tropic of Cancer and has an angle of 66.5 at the Equator and is at the horizon at the Antarctic Circle (66.5 S). Below the Antarctic Circle it is dark 24 hours a day. Above the Arctic Circle (66.6 N) there is sunlight 24 hours a day.

Equinox: 0 degrees (March 21 and September 22)

At the Spring and Autumnal Equinox, the sun is positioned over the Equator and day length is 12 hours everywhere on Earth.

Winter solstice: -23.45 degrees (December 22)
At winter solstice, the sun is directly overhead the Tropic of Cancer (23.5° N) at noon. It is dark 24 hours a day above the Arctic Circle (66.5° N) and it is bright below the Antarctic Circle for 24 hours.

The seasonal changes in the Earth-Sun geometry are summarized in Figure 8.

**Basic Earth-Sun Geometry**

![Figure 3.5 Earth's orbit and the Sun. Source: Lutgens/Ensback, *Atmosphere*, 6th Edition, 1995.](image)

A mathematical equation for computing the declination angle is:

\[ \delta = -23.45 \cdot \cos \left( \frac{360(D + 10)}{365} \right) \text{ (degrees)} \]

\[ \delta = -23.45 \frac{\pi}{180} \cdot \cos \left( \frac{2\pi(D + 10)}{365} \right) \text{ (radians)} \]
where $D$ is day number (after Jones (1995)). In this case the declination angle is expressed in terms of degrees. Multiplying by $\pi/180$ converts degrees to radians.

![Figure 9 Seasonal variation in the declination angle](image)

To understand how the declination angle varies with time, geometricians define a celestial sphere, of infinite radius, that rotates around the earth, the center point of the sphere. The intersection between the equatorial plane of the Earth with the celestial sphere defines the celestial equator. From the point of view of the celestial sphere, the sun follows on orbit over the course of a year on the ecliptic. It is tilted 23.5 with respect to the celestial equator.

The sun’s position on the celestial sphere can be estimated by knowing the angle of the right ascension and the declination. The right ascension is the angle between the hour circle of the vernal equinox and the hour circle of the sun (the angle between the sun and the north pole of the celestial sphere).

How azimuth angle is computed depends on the coordinate system which one uses. A mathematical perspective increments angles, starting from due south and rotating counter clockwise, from 0 to 360 degrees. A compass considers true north to be zero and increases angles in a clockwise manner from 0 to 360. An astronomical coordinate system increases azimuth angles from 0 to 180 in the clockwise direction and from 0 to –180 in the counter clockwise direction, starting with south as zero.

One compute azimuth angle in the mathematical system as a function of the declination and zenith angles and latitude (Campbell, Norman, 1998; Monteith, Unsworth, 1990):
\[
\cos(Az) = \frac{-\sin \delta - \cos \gamma \sin \lambda}{\cos \lambda \sin \gamma}
\]

(east is 90 degrees, north is 180 and west is 270 degrees).

Alternatively, one can compute solar azimuth, relative to south, as a function of the declination, hour and the solar zenith angles:

\[
\sin(Az) = \frac{-\cos \delta \sin(h')}{\sin \gamma}
\]

Here \(h'\) as the hour angle which the Earth has turned after local solar noon, as a fraction of \(2\pi\), \(h'=2\pi t/24\).

From (Oke, 1987):

If \(t < 12\)

\[
\cos(Az) = \frac{\sin \delta \cos \lambda - \cos \delta \sin \lambda \cosh}{\sin \gamma}
\]

If \(t > 12\)

\[
\cos(Az) = 2\pi - \frac{\sin \delta \cos \lambda - \cos \delta \sin \lambda \cosh}{\sin \gamma}
\]

An illustration of the motion of the sun at a particular location for several times during the year is presented in Figure 10. Of course the sun is the highest at noon and it rises in the east and sets in the west during the Equinox. For northerly climes, the sun is perceived to rise in the northeast and set in the northwest during the summer. And if we sit above the Arctic circle, the sun will revolve around the horizon during the summer.
We can manipulate this equation of the solar elevation angle to compute the times of sunrise and sunset, too.

The time of sunrise and sunset occur when the solar elevation angle is zero (or the zenith angle is $\pi/2$). It is a function of the latitude of the site and the Earth’s declination angle

$$\cos(h) = -\tan \lambda \tan \delta$$

If we convert $h$ to degrees, we can compute sunrise and sunset times:

$$t_{\text{sunrise}} = t_{12} - \frac{h}{15}$$

$$t_{\text{sunset}} = t_{12} + \frac{h}{15}$$

Based on the assumption that the sun moves 15 degrees per hour or 15 times $\pi/180$ radians.
The length of the day is $t_{\text{sunrise}} - t_{\text{sunset}}$, which equals $2h/15$.

$$2t = \frac{24}{\pi} \cos^{-1}(\tan \lambda \tan \delta)$$

Numerous algorithms are now available on the world-wide-web for computing solar elevation angles over the course of a day or year for any position on the Earth. One source of information on solar elevation angles is giving at:

http://susdesign.com/sunangle/

It has been updated since 2000 and is easy to use

**L7.4 View Factors**

View factors are important for they give us information on the projected area of various geometry shades (Monteith, Unsworth, 1990). The ratio of the area of shadow cast on the horizontal ($A_h$) and the area projected in the direction of the beam ($A$) yields information on the relation between the mean flux density and the horizontal flux density.

$$A_p S_p = (A_h \sin \beta) S_p = A_h S_b$$

If the area of the object is $A$ then

$$\overline{S}_b = \frac{A_h}{A} S_b$$

The shape factor is denoted by $A_h/A$. For the case of a sphere, we know that it projects a circle with an area of $\pi r^2$. The shadow projected onto the ground from a beam source is:

$$\frac{\pi r^2}{\sin \beta}$$

The surface area of a sphere is:

$$4\pi r^2$$

This yields the shape factor:
\[ \frac{A_h}{A} = \frac{\pi r^2}{4\pi^2 \sin \beta} = \frac{1}{4 \cos \gamma} \]

Figure 11 Geometry of a projected sphere

Cone:

\[ \frac{A_h}{A} = \frac{(\pi - \theta_o) \cos \alpha + \sin \alpha \cot \beta \sin \theta_o}{\pi (1 + \cos \alpha)} \]

Inclined Plane:

\[ A_h = \sin \alpha \cot \beta - \cos \alpha \]

Figure 12 Projection of inclined plane on a surface
L7.5 Radiation on inclined surfaces, mountain slopes

The Earth is not flat. Ecosystems exist on sides of hills that may face east, west, south or north. The amount of direct sunlight incident on the side of a hill will depend on the angle between the sun’s beam and the angle normal to the slope (Kondratyev, 1977).

Direct solar radiation flux density on an inclined surface

\[ R(i) = R_\perp \cos(i) \]

\( R(i) \) is the radiation normal to an inclined surface
\( R_\perp \) is the radiation flux density normal to the sun's rays

The angle of incidence on any given surface is computed as:

\[ \cos(i) = \cos \alpha \sin \beta + \sin \alpha \cos \beta \cos \psi \]

\( \alpha \) is the angle of the inclination of the surface relative to a horizontal plane.
\( \beta \) is the solar elevation angle
\( \psi \) is the difference between the sun's azimuth angle and the azimuth angle of the project normal to the surface on the horizontal.

Alternatively this equation can be expressed in terms of the sun's zenith angle (\( \gamma \)), where \( (\gamma + \beta) = \pi/2 \)

\[ \cos(i) = \cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos \psi \]

Angle between any surface and the sun:

\[ \cos \xi = \left[ \sin \lambda \cos(h)(-\cos \alpha \sin \chi) - \sin(h)(\sin \alpha \sin \chi) \right] + \left( \cos \lambda \cos(h) \right) \cos \delta + [\cos \lambda \cos \alpha \sin \chi] + \sin \lambda \cos \chi \sin \delta \]

\( \xi \) is the angle between the sun and the surface normal
\( \chi \) is the zenith angle of the slope
\( \alpha \) is the azimuth angle of the slope

**SUMMARY**
- Lamberts cosine law describes the amount of radiation received on the horizontal relative to the sun’s zenith angle.
- The amount of radiation received on an inclined surface is a function of the angle between the sun and the projection normal to the surface.
- The axis of rotation of Earth is tilted 23.5 degrees.
- The amount of radiation received is a function of its latitude and the sun’s declination angle.
- Seasons arise as the Earth revolves around the sun because the Earth’s tilted rotation axis causes the shaded and sunlit faces of Earth to change.

Figure 13 [Link to seasons image]

EndNote References


References/Bibliography:


Grebet, Ph. Functions subroutines and a program for easy computation of commonly required data related to the sun. In: *Crop Structure and Light Microclimate*. INRA eds. Varlet-Grancher et al.


Appendix

In spherical coordinates, we define three directional variables:

\[ z = r \cos \gamma \]
\[ x = r \sin \gamma \cos \phi \]
\[ y = r \sin \gamma \sin \phi \]

where \( \gamma \) is the zenith angle and \( \phi \) is the azimuth angle. These angles are for an Earth-centered perspective.

In Vector notation we define:

\[ \vec{S} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r_s \sin \gamma \cos \varphi_s \\ r_s \sin \gamma \sin \varphi_s \\ r_c \cos \gamma \end{bmatrix} \]
\[ \vec{n} = \begin{bmatrix} r_e \sin \gamma_e \cos \varphi_e \\ r_e \sin \gamma_e \sin \varphi_e \\ r_e \cos \gamma_e \end{bmatrix} \]

Remember:

\[ \vec{a} \cdot \vec{S} = a_xS_x + a_yS_y + a_zS_z \]
\[ \vec{n} \cdot \vec{S} = \vec{S} \cdot \vec{n} \]
\[ |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} \]

and a triangle with vertices at A, B and C:

\[ \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC} \]