APPENDIX

The Assimilation of the Solution

The remaining historical question is that of the period of assimilation of the acceptance of both mv and mv^2 into the general body of physical knowledge. How long for example did it take before problems of elastic impact were handled by the simultaneous application of both conservation equations? Nowhere does d'Alembert set up the simultaneous solution of the two equations as a model for the general method of attacking elastic collisions exemplifying the validity of both principles and the resulting simplicity. For the investigation of this question we must turn to general textbooks, lecture collections, mathematical dictionaries, and encyclopedias. Many of the works investigated make no mention of the problem. But those authors who do treat the problem of collision were found to employ only the conservation of momentum. A brief sampling and chronological summary of their methods of approach follows.

In A Course of Lectures in Natural Philosophy (1743), Richard Helsham of the University of Dublin discusses non-elastic and elastic collisions. For each he gives one general proposition followed by rules for particular cases and experimental confirmation. His proposition for non-elastic bodies is:

> If two bodies void of elasticity move in one right line, either the same or contrary ways, so as that one body may strike directly against the other; let the sum of their motions before the stroke when they move the same way and the difference of their motions when they move contrary ways, be divided into two such equal parts as are proportional to the quantities of matter in the bodies; and each of those parts will respectively exhibit the motion of each body after the stroke.[1]

He cites four general cases: (1) one body is in motion at the time of the stroke, (2) both are moving in the same direction, (3) both move in opposite directions with equal quantities of motion, (4) both move with unequal motions in contrary directions.

The proposition for elastic bodies is:

> If of two bodies perfectly elastic, one be at rest, and the other in motion; or if they both move either the same or contrary ways, so as that one shall strike the other; let them be considered as void of elasticity and by the proposition laid down in my last lecture let the motion of each body after the stroke be found, and by one of the four rules laid down in the same lecture, let the motion communicated by the striking body to the other be likewise found, and let this motion be subducted from the motion of the striking body after the stroke, and added to that of the body which received the stroke and the residue will be the motion of the striking body, and the sum the motion of the other body after reflection.[2]

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[1] Richard Helsham, <u>A Course of Lectures in Natural Philosophy</u>, London, 1743, 58.

[2] Helsham, 68.

The various cases are illustrated by collisions between balls of clay and ivory suspended from threads, the heights of descent and rise being measured.

An anonymous author of La Physique experimentale et raisonnée likewise formulates separate rules for the several individual cases of both elastic and inelastic collisions. For example,[3] the change produced by a hard elastic body hitting another at rest is found by (1) dividing the quantity of motion of the striking body by the sum of the masses of the two bodies. (2) Divide the respective velocity (i.e. the resultant of the two velocities at the time of collision) into two parts reciprocally as the masses. (3) Add similar motions and subtract contrary ones. Thus a body of mass 3, velocity 4 /designated 3M(4)7 hitting a body of mass 3 at rest /3M(0) 7 will produce the result: step (1) $\frac{mv}{m_1 + m_2} = \frac{12}{6} = 2$. (2) The respective velocity, 4, is divided into two equal parts because the masses are equal. (3) The results of steps (1) and (2) are added and subtracted, giving the conclusion that body A rests after the collision and body B receives 4 degrees of velocity. In the author's shorthand notation the problem is set up as follows, with the dash indicating the direction of motion):

	3M4-	3M0
(1)	3M2-	3M2-
(2)	-3M2	3M2-
(3)	3M0	3M4-

[3] La Physique experimentale et raisonnée, Paris, 1756, 16-18.

Evidence that the resolution of the controversy over living forces had not been accepted or widely distributed in Europe is derived from a belated contribution to the debate. Offered as late as 1772, was Abbé Para du Phanjas Theorie des etres sensibles ou Cours complet de physique. This work discusses collisions in terms of momentum and gives a long refutation of Leibniz's living forces.[4]

The first indication that both conservation laws are recognized as correct is found in Diderot's encyclopedia of 1778. However since the article was written by d'Alembert it cannot be regarded as a reflection of the true progress of assimilation. He writes that the resolution of the controversy as a dispute over words has obtained the almost unanimous agreement of all geometers.[5]

Later analytic formulations of the *vis viva* concept are given by LaGrange in his *Mécanique analytique* (1788) and LaPlace in his *Mécanique celeste* (1799). These give emphasis to central forces derivable from potential functions, using the integral of the force times the differential path element. The law equates *vis viva* with a function of position coordinates. These discussions however do not refer

[4]Abbé Para du Phanjas, *Theorie des etres sensibles ou Cours complet de physique*, Paris, 1772, 304-322.

[5]Denis Diderot, *Encyclopedie ou dictionaire raisonné des science, des arts et des metiers; Partie mathématique par M. d'Alembert*, Geneva, 1778, 41, 958.

to impact problems.

Like the two textbooks mentioned above Mathieu's Nouveau systême de l'Univers (1796) gives a set of general rules for elastic and non-elastic collisions. If a soft body hits a body at rest the velocity divides itself between the two in accordance with their masses. If they move in the same direction the velocity of the striking body is diminished and that of the other body augmented by as much. If they move in contrary directions, the two follow the direction of the strongest by a velocity proportional to the excess of its motion over that of the weaker.[6]

The rules for elastic collisions are of the same general type as given by the preceding authors. Mathieu nowhere mentions mv^2; he says that the laws of matter and their effects given in his book are the totality known of nature's laws.[7]

A second scientist to mention the validity of Leibniz's principle of living force and who used the term "energy" was Thomas Young. In a chapter "On Collision" in his book, A Course of Lecture on Natural Philosophy and the Mechanical

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[6]Charles Leopold Mathieu, Nouveau systême de l'Univers, Paris, 1796, 100.

[7]Mathieu, 101.

Arts (1807) he writes:

> The term energy may be applied with great propriety to the product of the mass or weight of a body, into the square of the number expressing its velocity.... This product has been denominated the living force or ascending force, since the height is in proportion to it; and some have considered it as the true measure of the quantity of motion; but although this opinion has been very universally rejected, yet the force thus estimated well deserves a distinct denomination.... Leibniz, Smeaton, and many others have chosen to estimate the force of a moving body by the product of its mass into the square of its velocity; and although we cannot admit that this estimation of force is just, yet it may be allowed that many of the sensible effects of motion...are usually proportional to this product.
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> ....
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> An elastic ball of two ounces weight moving with a velocity of three feet in a second, possesses an energy as we have already seen, which may be expressed by 18. If it strike a ball of one ounce which is at rest, its velocity will be reduced to one foot in a second and the smaller ball will receive a velocity of four feet; the energy of the first ball will then be expressed by 2 and that of the second by 16 making together 18 as before. The momentum of the larger ball after collision is two, that of the smaller four, and the sum of these is equal to the original momentum of the first ball.[8]

Although Young admits the utility of both the momentum and energy principles, he does not suggest their simultaneous application as the most general method of attacking collision problems.

In 1824, William Whewell mentioned the validity of both mv and mv^2 in his discussion of impact. Proposition 199 says: "In the direct impact of elastic bodies the sum of each body into the square of its velocity is the same

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[8]Thomas Young, A Course of Lectures on Natural Philosophy and the Mechanical Arts, London, 1807, I, 78-80.

before and after impact.

$$Aa + Bb = Au + Bv$$

$$A(a-u) = B(v-b)$$

also: $$a-b + v-u \text{ or } a^{+}u = v+b$$

hence by multiplying the [two] equations:

$$A(a^2 - u^2) = B(v^2 - b^2)$$

$$Aa^2 + Bb^2 = Au^2 + Bv^2$$ [9]

Olmsted's Compendium of Natural Philosophy (1833) contributes no new approach to the problem. He discusses the question in terms of two generalizations. Concerning conservation of momentum, he writes, "It is a general law in the material world that no body loses motion in any direction, without communicating an equal quantity to other bodies in that same direction, and conversely that no body acquires motion in any direction, without diminishing the motion of other bodies by an equal quantity in that same direction....Hence it follows, that the sum of the motions of all the bodies in the world, estimated in one and the same line of direction, and always the same way is eternally and invariably the same."[10]

The second generalization he makes is, "In the collision of perfectly elastic bodies, the velocity lost by

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[9] William Whewell, Elementary Treatise on Mechanics, Cambridge, 1824, 258.

[10] Denison Olmsted, A Compendium of Natural Philosophy, New Haven, 1833, 25-26.

the one and gained by the other, is twice that which it would have been, had they been perfectly non-elastic".[11] This statement is not demonstrated by the author however.

The First Principles of Natural Philosophy (1842) by James Renwick spells out in detail the same general rules for calculation as given a century earlier by the authors already mentioned.[12]

Thus almost one-hundred years after d'Alembert and 150 years after Huygens and Leibniz had discovered the basis for a general solution to the impact of elastib bodies, the problem was still being treated by a series of empirical rules in terms of only one conservation equation.

It thus seems probable that the more general method of attack had to await the fuller understanding and development of energy relations which took place during the nineteenth century. After the development of the first law of thermodynamics by Mayer, Joule, and Helmholz and Colding in the 1840's, the universality of the concept of energy became widely recognized. The full significance of the conservation of both kinetic energy and of momentum for impact situations was understood at least by the 1860's. In The Circle of the Sciences (1862-67)[13] the problem of elastic

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[11]Olmsted, 25.

[12]James Renwick, First Principles of Natural Philosophy New York, 1842, pp. 68-70.

[13]James Wylde, The Circle of the Sciences, London, 1862-67, 1, 745.

collisions according to the solution of two simultaneous equations, conservation of momentum and of energy. Thus nearly two-hundred years after Leibniz had first solved impact problems by these two simultaneous equations, the method is assimilated into general physics textbooks.