CHAPTER VIII

Jean Bernoulli and the French Reactions (1727-1729)

The paper submitted by the Swiss mathematician Jean (John or Johann) Bernoulli, professor at Basil, for the 1724 contest was disqualified by the French Academy on the grounds that Bernoulli did not discuss the impact of "hard" bodies in the sense meant by the Academy. He had rejected the existence of hard bodies in the sense of inflexible, unbreakable atoms and took hardness to pertain to bodies having perfect elasticity under infinite internal pressure. The Academy had responded to this interpretation by disqualifying Bernoulli and announcing another contest dealing with the laws of impact for elastic bodies to be held in 1726. For this contest Bernoulli resubmitted his paper of 1724, "Discours sur les loix de la communication du mouvement" adding an appendix containing a probable explanation of the physical cause of elasicity.[1]

This time although the winner was Pierre Mazière, Bernoulli's paper was again awarded honorable mention and

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[1] Jean Bernoulli, "Discours sur les loix de la communication du mouvement" in the Recueil des pieces qui a remporté les prix de l'academie royale des sciences, 2 (1727), 1-108, separate pagination.

printed in the Academy's collection of prize winning essays.

Following Leibniz,[2] and basing his argument on Leibniz's law of continuity, Bernoulli rejected in his "Discourse" the existence of hard bodies in nature.[3] Since every act occurs by infinitely small degrees and "nature does not operate through leaps," motion cannot pass suddenly into rest, or rest into motion as would be necessary in the collision of two hard bodies.[4] Hard bodies being inflexible and unbreakable, they would not rebound after colliding, their speed dropping to zero without going through intermediate steps. If this were true there would be no reason why nature would choose one state of motion or rest in preference to another, since having no liasion between the two states, rest to motion or motion to rest, no reason would determine the production of one over the other.[5]

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[2]Through his correspondance with Leibniz, Bernoulli had first become convinced of Leibniz's views on the conservation of force, mv^2. See Leibniz, Commercium philosophicum et mathematicum virorum celebr, Got. Gul. Leibnitii et John Bernoulli, Lausanne, 1745, 2 vols.

[3]For a valuable discussion of the papers of Maclaurin, Bernoulli and Mazière from the point of view of hard bodies see Wilson Scott, The Significance of Hard Bodies in the History of Scientific Thought, John Hopkins University,1960. This work is, however, significant in completely ignoring the contributions of Leibniz to the question of vis viva and the heavy dependence of Bernoulli on Leibniz's ideas.

[4]Bernoulli, op. cit., 5.

[5]Ibid., 5.

Hardness taken in the common sense of perfectly solid atoms is rejected, these atoms being imaginary corpuscles existing only in the minds of their champions.[6]

Bernoulli asserted that hardness existed only in the sense that bodies are like heavy "balloons filled with compressed air." The greater the pressure the harder the surface but likewise the more perfect the body's elasticity. If the density of the air in the balloon is increased to an immense degree of resistance such that an extremely powerful force is necessary to compress it, the balloon differs in no essential aspect from a hard body.

If one imagines a number of small balloons full of extremely condensed air in a common envelope, then one can give meaning to "hardness" in bodies. The small balloons represent elementary molecules and the envelopes take the place of an ambient fluid which by its own activity presses and compresses the entire mass. If an infinitely large degree of elasticity is contained in these balloons, their entire mass cannot be sensibly compressed by "a finite force as large as can be supposed."[7]

"A body will conform to our idea of hardness when its sensible parts change their situation only with dif-

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[6] *Ibid.*, 6.

[7] *Ibid.*, 9.

ficulty." "Elasticity is perfect when all the parts return to their original state; it is imperfect when some of the parts do not return."[8] This then is the meaning of "hard body" on which the laws of motion developed in the remainder of the essay are based.

Before discussing the measure of force and again taking his conceptions from Leibniz, Bernoulli describes what he means by living and by dead force.

"*Force vive* or living force is that which resides in a body when it is in uniform motion. *Force morte*, or dead force is that which a body not in motion receives, when sollicited or pressed toward motion," or which moves it more or less fast when the body is already in motion.[9]

For example if an obstacle prevents local motion from occuring in a body, the body has dead force. The force of gravity is another example: A body placed on a horizontal table makes a continual effort to descend. At each instant gravity imprints on a body on which it acts, an infinitely small degree of velocity which is immediately absorbed by the resistance of the obstacle. "These small degrees of velocity perish on creation and are reborn in perishing." It is this constant reciprocation in the recurring production and destruction in which the force of gravity consists, when acting on an invincible object,

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[8] *Ibid.*, 9.

[9] *Ibid.*, 19.

that is called dead force.

The nature of living force is totally different. It is not born and does not perish in an instant as does dead force. Time is needed to produce living force and likewise to destroy it. Living force is produced successively in a body as a pressure applied to the body imprints little by little, degrees of local motion. Motion is acquired by infinitely small degrees, becoming finite and determined and finally remaining uniform when the cause which produced it ceases to act on the body. Thus living force is produced in a finite time by a pressure and is equivalent to that part of the cause which is consumed in producing it.[10]

The living force of a body produced by the dilation of some elastic body is capable of compressing it again to precisely the same state in which it was originally. There is complete equality between the efficient cause and the effect. In this equality consists the conservation of force of bodies in motion.

Since it has been believed for a long time that quantity of motion or the product of the mass of a body by its velocity is the measure of the force of this body, it was falsely believed that it was necessary for there always to be an equal quantity of motion in the universe.

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[10]Ibid., 32, 33.

Following Leibniz's opinion Bernoulli attributes this error to the confusion of dead forces with living forces. Seeing that the fundamental principle of statics lies in the equilibrium of "powers", the moments are composed of absolute forces and their virtual velocities. In extending this principle to the forces of bodies which have actual velocities, philosophers have gone too far.[11]

In this essay Bernoulli defined virtual velocities in the following way:

> I call virtual velocities /vitesses virtuelles/ those acquired by two or more forces taken in equilibrium when a small movement is imprinted upon them; or if these forces are already in motion. The virtual velocity is the element of velocity already acquired that each body gains or loses in an infinitely small time along its direction.[12]

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[11] Ibid., 35. For Leibniz's analysis of the confusion over dead forces see this dissertation, Ch. III.

[12] Ibid., 19. Bernoulli had first defined the principle of virtual velocities in a letter to Pierre Varignon in 1717: (See Erwin Hiebert., op. cit., 82)

> Imagine several different forces which are acting along different tendencies or directions to maintain at equilibrium a point, a line, a surface, or a body; imagine also that we impress on the whole system of forces a small movement either parallel to itself along any direction, or about any fixed point: it is easy to see that by this movement each of these forces will advance or recede in its direction, unless some one or more of the forces have their directions perpendicular to the direction of the small movement; in which case the force or the forces will neither advance nor recede: for these advancements or recessions which I call virtual velocities, are nothing other than the amounts by which each line of tendency increases or decreases because of the small movement; and these increments or decrements are found by drawing a perpendicular from the end of each line of tendency

Bernoulli who considered himself the foremost champion of living forces following Leibniz's death, believed that Leibniz's demonstration of 1686 was not convincing. In the succeeding chapter of his "Discourse" he presented proofs of the measure of living forces, such as the following which he regarded as incontestable.

He first developed some preliminary principles necessary to his argument.

1. An elastic body ABC which is held in a state of compression by one or more forces is in equilibrium with those forces. (Figure 3, p. 282.)
2. If an elastic body is held in compression by 2 powers, these two powers are equal. (Figure 4, p. 282.)
3. If an elastic body is held in compression by

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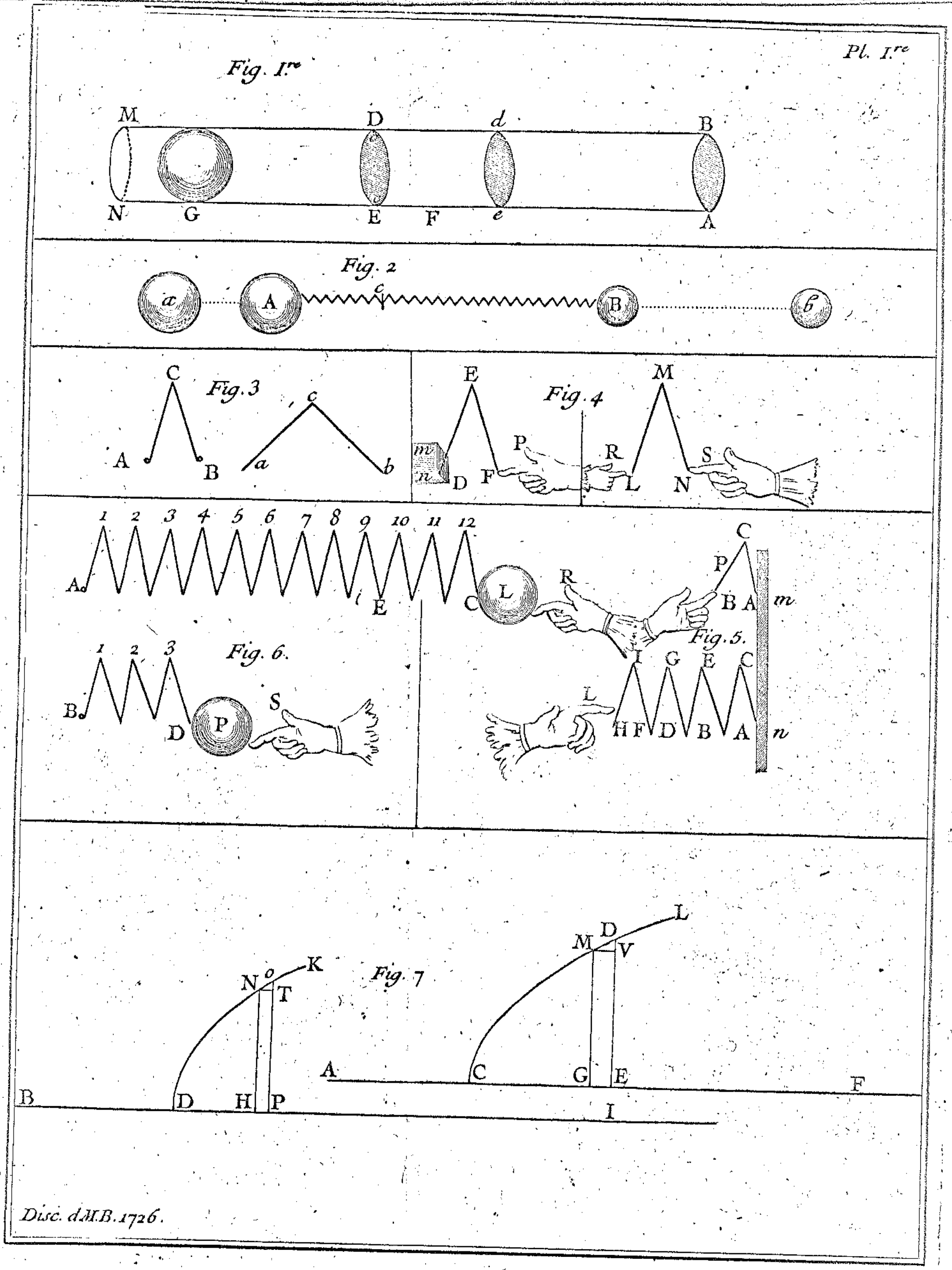
of each force in the neighboring position, to which it has been brought by the small movement, a small portion of which will be the measure of the virtual velocity of this force.

(Quoted in Pierre Varignon, Nouvelle mécanique ou statique Paris, 1725, 2, 174.) Bernoulli followed this with a definition of virtual work which he called energy, that is the product of the displacement and the force producing that displacement: (See Hiebert, 83-84)

In every equilibrium of forces whatsoever, in whatever way they may be applied, and in whatever directions they act on one another, either mediately or immediately, the sum of the positive energies will be equal to the sum of the negative energies taken positively. (In Varignon., 175-176).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two powers R and S, and one substitutes in place of power R, an immovable plane mn, the power S will make no greater effort than before. (Figure 4, p. 282).

4. If two powers P, L hold several equal elastic bands in a state of compression they will make no more effort than if they compressed only one elastic, ACB. (Figure 5, p. 282). This is demonstrated by the preceding rules since the intervening bands press equally against each other.[13]

Consider two groups of elastic springs composed in the ratio of 12 units to 3 units (figure 6, p. 282). One of the extremities of each is held at the fixed points A and B; the other is fastened to the equal balls L and P, held in compression by powers R and S. The springs are equally compressed so that the "dead forces" (tendencies toward motion) in the balls are equal. When the powers, R and S, are removed, the dead forces produce living forces as the springs expand, causing the balls to accelerate away from A and B. Ball L acquires more velocity than ball P because the forces of the elastics are in the ratio of 12 to 3.

[13] Ibid., 39, 40.

To express the results mathematically, consider two straight lines AC and BD in the ratio of 12 to 3, representing the units or coils of the two elastic springs (figure 7, p. 282).

When the elastics expand the balls begin to move from C and D toward the points F and I. The ordinates GM, HN of the curves CML and DNK represent the velocities acquired at the points G and H. Let BD = a, the distance DH = abcissa x, increment HP = dx, the ordinate HN = u and its increment TO = du. Take BD : AC :: DH : CG : : DP : CE and suppose AC = na, CG = nx, GE = ndx, ordinate GM = z, increment DV = dz.

Since AC : CG :: BD : DH, at each instant each of the elastic springs has lost an equal part of its force so that the dead forces and pressures of the balls at G and H are equal. The increment of velocity H,(du) which is in accordance with the measure of "dead forces" or pressure is composed of this pressure, p, and the increment of time in which the moving body traverses dx. Thus du = pdt.[14] Since the velocity, u, at any instant is expressed as $\frac{dx}{dt}$, the increment $dt = \frac{dx}{u}$. Thus we have $du = \frac{pdx}{u}$ or $udu = pdx$ of which the integral is $1/2\ uu = \int pdx$. By the same reasoning $dz = \frac{p(ndx)}{z}$, and by integration $1/2\ zz = \int pndx$.

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[14] This definition of "dead force" will be challenged later by Louville (1729).

From this it follows that $\underline{uu} : \underline{zz} :: \int \underline{pdx} : \int \underline{pndx} :: \underline{a} : \underline{na} :: \underline{BD} : \underline{AC}$. But BD is to AC as the living force acquired at H is to the living force acquired at G, or from the above, as uu is to zz. Thus Bernoulli has proved that the living forces in equal bodies of equal masses are as the squares of their velocities.[15]

As a corollary to this proof, if bodies are of unequal masses, their living forces are as the products of the masses by the squares of their velocities.

Thus the acceleration of the bodies in this case follows the same law as that of falling bodies where the squares of the velocities acquired are as the heights traversed in the fall of these bodies.[16]

Bernoulli's succeeding chapter confirms this measure of force by experiments. Here experiments such as those of 's Gravesande and Poleni are described, allowing different balls to fall into soft clay from various heights and measuring the depth of the impressions so formed.[17]

A second argument of Bernoulli was thought by him to be completely general and capable of convincing even the most obstinate of partisans of the commonly held opinion

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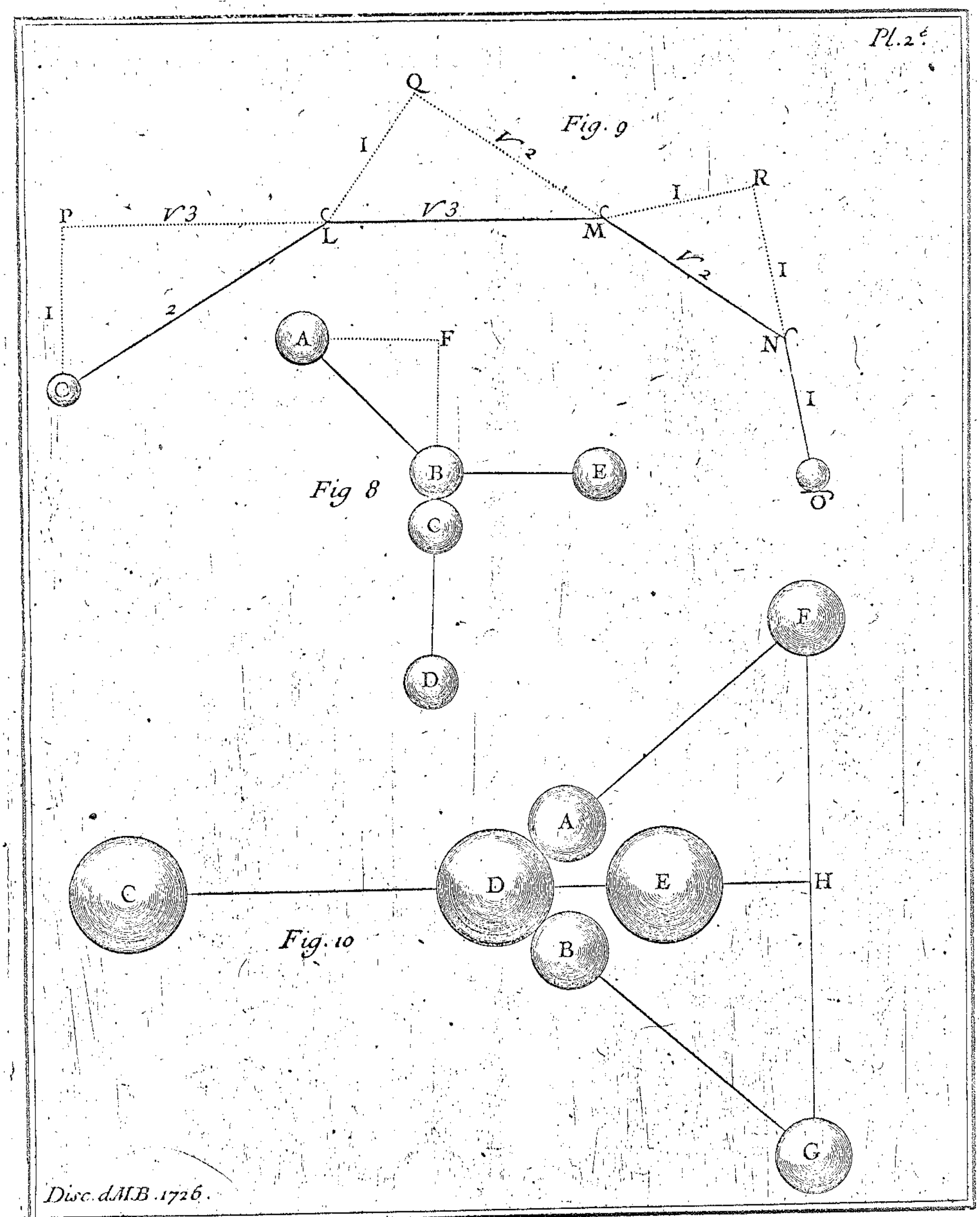
[15] *Ibid.*, 41-45.

[16] *Ibid.*, 45, 46.

[17] *Ibid.*, 48, 49.

favoring mv as the measure of force.

Suppose, says Bernoulli, that a body C moving in the horizontal plane strikes obliquely an elastic band placed at L with the velocity CL = 2. (See figure 9, p. 287). The angle of obliquity CLP is 30° and the perpendicular CP is equal to $\frac{1}{2}$ CL. The resistance of the elastic band L is such that in order to bend it, body C must have 1 unit of velocity when striking it perpendicularly. Body C striking L obliquely with velocity 2 has a motion composed of a perpendicular component CP = 1 and a second component PL $=\sqrt{3}$. In striking L it loses its perpendicular component, retaining only the component PL $=\sqrt{3}$. The body thus continues to move only in the direction PLM with velocity LM = PL $=\sqrt{3}$. At point M there is a second elastic band ressembling the first with angle of obliquity LMQ such that the perpendicular LQ = 1. Body C will then continue by the same argument as before along direction QMN with velocity MN equal to QM $=\sqrt{2}$. At point N is a third elastic band equal to the other two such that the body collides at $\angle$ MNR = 45°. Component of motion MR, perpendicular to the line of the elastic band is equal to 1. The motion MN composed of motions MR and RN will lose the component MR in bending the elastic N. Body C will thus continue with velocity NO = RN = 1, having bent the 3 elastic bands L, M, N. With this one degree of velocity it bends the 4th band O, against which it collides perpendicularly. Body C having 2 degrees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of velocity, has now bent 4 elastics, each requiring for bending, one degree of velocity. These 4 bent elastics represent the total force with which body C has finally consumed its motion. But the effects are proportional to the forces which have produced them. Thus the living force of body C having 2 degrees of velocity is four times greater than the living force of the same body having one degree of velocity.

In the same manner it is demonstrated that a velocity triple, quadruple, quintuple etc. gives to body C a force 9, 16, 25 etc. times as great. From this is drawn the general conclusion that the living force of a body is proportional to the square of its velocity and not to its simple velocity.[18]

Bernoulli recognizes in this "Discourse" the same three conservation laws for the collision of bodies developed by Huygens and placed in their proper significance by Leibniz. The first is the conservation of the same respective velocity before and after collision, $a-b=y-x$ where a and b are the velocities of bodies A and B before collision and y and x their velocities after collision, and where the bodies both proceed in the same direction before and after colliding. The respective velocity is the difference of the absolute velocities when the bodies move

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[18] *Ibid.*, 51-53.

in the same direction, the sum of the absolute velocities when their directions are contrary. The second law is the conservation of the quantity of direction and is equal to the product of the sum of the masses by the velocity from the common center of gravity of the two bodies, Aa + Bb = Ax + By. Bernoulli did not state the fact that this was conservation of momentum with the sign taken into consideration. Like Leibniz he thought of the principle as refuting Descartes' concept of the quantity of motion.

> From this it appears that the quantity of motion is not always conserved as commonly imagined. And in effect this quantity is conserved only in two cases, 1) when the body moves from the same side before and after collision: 2) when the quantity of direction is null, or the common center of gravity is without motion, because then the bodies each reflect with their original velocities.[19]

The third law is the conservation of the quantity of living forces and can be derived from the first two laws Aaa + Bbb = Axx = Byy. This expresses something existing in bodies which move that is absolute, independent, and positive which remains in the body and cannot be annihilated in the universe. The "force vive" of a body diminishes or augments when it encounters other bodies and the "force vive" of this other body is augmented or diminished by the same quantity. The increase of one is the immediate effect of the decrease of the other. This quantity is absolutely unalterable by the collision of bodies.

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[19] Ibid., 29.

Bernoulli thus followed Leibniz almost to the letter, but did not discuss in this paper the problem of *force vive* in inelastic collisions. His own contribution, he believed, was to have established the truth of living forces in a manner so evident as to be incontestable.[20]

Of the foregoing papers Bernoulli's caused the greatest stir in the French intellectual world. Most of the subsequent papers written for the French Academy on the subject of the force of bodies in motion are in some way a reaction to Bernoulli's proof of *vis viva* using elastic springs. Thus the physical problem of the properties of these springs with respect to momentum and *vis viva* became of great importance.

Bernoulli's concept of matter based on the law of continuity and his rejection of hard atoms signifies the important role which conservation of force played in his scientific viewpoint. However, the two *vis viva* proofs outlined above are only concerned with establishing its validity as a measure of force. In these proofs he does not claim conservation of force, since the forces of the bodies are merely compared and the bodies do not interact. The validity of conservation is, however, claimed in his

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[20] *Ibid.*, 53-55.

description of impact problems (as well as in numerous problems in statics). In this essay Bernoulli did not discuss the problem of inelastic impact. According to Erwin Hiebert:

> In a treatise of 1735 on the nature of living force, he treated living force, more fittingly called, le pouvoir, as something substantial, existing by itself and by its quantity, and depending on nothing else. From this he concluded that any living force, vis viva, possesses a definite quantity none of which can perish without producing an effect. This force, he maintained, has its seat in one or several bodies before a process, and must of necessity be found in one or more other bodies after the process. For Bernoulli this was the essential meaning of the law of conservation of living force; conservatio virum vivarum. He therefore maintained as Leibniz had done, that wherever vis viva seems to disappear, the power to do work, facultas agendi, is not lost but is only changed into some other form.[21]

Concerning the issue of conservation and its relation to the "hard body" controversy in the early 18th century Wilson Scott writes:

> Defenders of MV wished to explain communication of force without violating the laws of cause and effect of which the third law of Newton is the scientific expression. The defenders of MV^2 in turn wished to explain communication of force in terms of conservation of force in nature so as to account for a stable universe.... But the MV^2 school con c eived the universe not in material but in dynamic terms, and were therefore holding to a dynamic conservation even if it meant the denial of hard bodies.
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> It is clear from Maclaurin's treatment that force cannot be conserved in the choc of hard bodies. Furthermore in the choc of soft inelastic bodies,

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21 Hiebert., op. cit., 84-85, and Johannes Bernoulli, "De vera notione virium vivarum earumque usu in dynamics, ostenso per exemplum," propositum in Comment. Petropolit. 2, 2, 200.

> the conservation principle has to be restricted to conservation in the same direction only.... This limitation on a basic principle of nature was annoying to those desiring conservation. But this annoyance could readily be dispelled by the algebraic trick of squaring the v, thus making direction irrelevant. Hence we have MV^2 advanced as the true measure of force... in a universe of elastic bodies which is obviously stable.[22]

Thus John Bernoulli together with 's Gravesande provided the main support for the Leibnizian measure of force. Bernoulli however did not base his discussion on the inelastic case of impact as had 's Gravesande. Bernoulli relied heavily on the *conservation of vis viva* in elastic impact; we have seen that he agreed with Leibniz that the *vis viva* which apparently disappeared in inelastic impact was really only changed into another form and still available to do work.

Just as 's Gravesande's 1722 paper had touched off a series of refutations and counter-experiments, so Bernoulli's "Discourse" inspired a series of essays examining his opinions. They were written mainly in response to Bernoulli's elastic-spring demonstration. The writers included Abbé Camus (1728), de Louville (1729), and Jean Jaques de Mairan (1728).

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[22]Wilson Scott, *The Significance of Hard Bodies in the history of Scientific Thought*, Thesis, John Hopkins University, 1960, 25-27. See discussion of Maclaurin, this dissertation, Ch. VII.

In an article "On Accelerated Motion due to Sorings and the Forces Residing in Moving Bodies"" (1728), Abbé Charles Etienne Camus[23] (1699-1768) established the relationships between the force of rising or falling bodies and the force of compression or expansion of springs. He showed that the forces of these accelerated motions were proportional to the masses of the bodies accelerated by gravity or pushed by compressed springs, and the squares of their velocities. This paper is important in understanding the many examples involving compressed springs used by other authors.

Camus defined an elastic spring as "a body which after having been distorted [or compressed] re-established itself nearly or exactly in the state in which it was before compression." A spring is perfectly elastic if "in re-establishing its state before compression, it gives back to the body distorting it all the degrees of velocity lost by that body."[24] A spring is imperfectly elastic if it does not return all the velocity to the compressing body. Springs with similar elasticity (ressorts semblables) are those whose resistances or forces (roideur) are always similar with respect to their aperatures. If for example

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[23] Abbé Charles Etienne Camus, "Du Mouvement accéléré par des ressorts, et les forces qui résident dans les corps en mouvement," Histoire de l'Academie royale des sciences, (1728), M., 159-196.

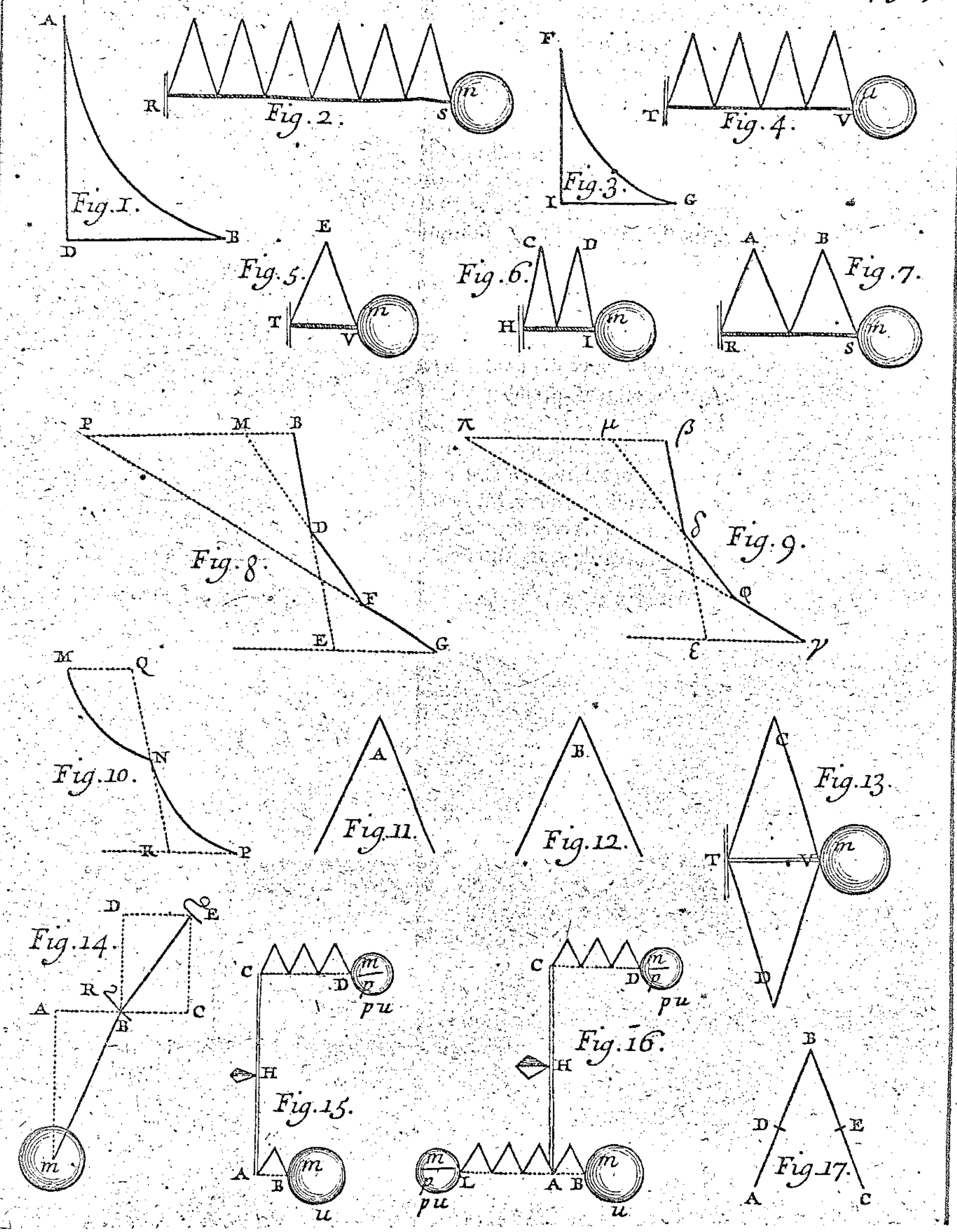
[24] Ibid., 159.

two springs A and B are such that the resistance or initial force (roideur) of spring A when it is compressed is to the resistance or initial force of spring B when it is compressed, as the resistance or force of spring A when it is open or held at an aperature of 15° (see diagrams, p. 295) is to the resistance or force of spring B when it is also open or retained at an aperature of 15°, then springs A and B are similar.[25]

Camus then derives the laws of motion pertaining to the compression and expansion of these similar springs when hit by other bodies.

Let there be taken a curve AB (see figures 1, 2) of the same length as the set of elastic springs RS which must be completely compressable and suppose the resistance or force of the spring RS when closed equal to the resistance that ascending body m finds due to its gravity at the summit A of curve AB. One conceives the curve AB such that the resistances that body m will find at different points of its ascent will be equal to the resistances which it will find at the corresponding points of space necessary to traverse in compressing spring RS. Since the curve AB = RS, and the resistances are distributed in the same manner along the curve AB and the space RS, it is clear that body m which rises along curve AB will be able with the same velocity

[25] Ibid., 159.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and in the same time to close spring RS.

If a second elastic spring TV is taken similar to RS (see figures 3,4) one can compare the variation in resistances that a body μ will find in compressing TV to the variation in resistance one finds in ascending curve FG = TV. Since the curve FG = TV, if body μ ascends curve FG, it can with the same velocity close the spring TV. But the resistances that body m will find in compressing the spring RS will always be in the same ratio as the resistances that body μ finds in compressing spring TV, in similar aperatures because these two springs are similar.

It is thus necessary that the resistances that body m finds in mounting curve BA be in the same ratio as the resistances of body μ mounting curve GF.

Consequently the masses m, μ will acquire in descending the curves AB, FG, velocities equal to those they will receive in the expansion of curves RS, TV. And the times that these bodies will employ in descending these curves will be equal to the times that elastics RS, TV will use to push masses m, μ in expanding. Further one supposes that the masses m, μ will receive at the points A and F of these curves, forces f, φ equal to those they will receive from springs RS, TV when they are compressed and begin to expand. The laws of accelerated motion along curves AB, FG will then also be the laws of accelerated motion of the similar springs RS, TV.[26]

[26] *Ibid.*, 160-162.

Assigning the following notation:

m, μ, the masses accelerated by the similar springs RS, TV.

f, φ, the initial forces of springs RS, TV when they are compressed.

u, v, the velocities that m, μ acquire in the expansion of RS, TV and hence the velocites they should have to compress these springs.

e, ϵ, the lengths of the spaces RS, TV.

t, θ, the times that springs RS, TV use to expand.

"One obtains four formulas A, B, C, D of accelerated motion of elastic springs or sets of similar springs, [demonstrating] that the products of their absolute size (grandeur absolue) and the sum of the obstacles that the bodies in motion can surmount are always as the masses of these bodies multiplied by the squares of their velocities."

A. $f\,t\,t\,\mu\,\epsilon = \varphi\,\theta\,\theta\,m\,e$

B. $f\,e\,\mu\,v\,v = \varphi\,\epsilon\,m\,u\,u$

C. $f\,t\,\mu\,v = \varphi\,\theta\,m\,u$

D. $\epsilon\,u\,t = e\,v\,\theta$ [27]

From these fundamental relations Camus proves many theorems. One important example is:

$$e : \epsilon :: m\,u\,u : \mu\,v\,v$$

That is, the length of the spaces RS, TV, or the number of

[27] Ibid., 169.

elastic bands composing the spring are in a ratio composed of the masses $\underline{m}, \mu$ and the squares of their velocities. This, says Camus, was also proved by Bernoulli in number 2 of his first hypothesis and in his corollaries.[28]

Another example directly involving conservation is as follows: "Let there be two perfectly elastic bodies $\underline{m}, \mu$. Let $\mu = 3m - 2m\sqrt{2}$ and be at rest. If the body $\underline{m}$ has a velocity sufficient to compress a set of two springs, I say that the body $\underline{m}$, in hitting directly the body μ that I have supposed at rest will communicate to it a velocity with which it can compress one of the springs of the set and that this body $\underline{m}$ will conserve again enough velocity to compress the second spring."[29]

Abbé Camus then defined the limits of the meaning of "Forces vives." He distinguished three different methods of estimating the "force of bodies in motion." He first discussed dead force. One can consider a body in motion at a single indivisible instant, estimated by the pressure or effort that it makes against an invincible resistance. The resistance destroys this effort in that instant by opposing to it a resistance equal to the effort made against it.

Hard bodies in motion which have velocities reciprocal

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[28] Ibid., 173, and 177.

[29] Ibid., 184.

to their masses such that their equal quantities of motion cause equilibrium between them, reciprocally make invincible resistances for each other during a single instant. Thus these forces residing in these bodies in motion at each instant are equal when the bodies have equal quantities of motion.

But the force of a body so considered is not properly the force of a body in motion since this force acts at an instant during which there is no traversal of space, and there is never motion unless space is traversed. This force which tends to produce motion but does not effect traversal of space conforms to the definition of dead force, or *force morte*. Dead force then is that by which a body is pressed and solicited to move itself, without actually moving. Camus gives the familiar example of the pressure of a body on a horizontal table as dead force because the body *tends* to traverse a space without actually moving, owing to the resistance opposed by the table. The measure of the dead force of a body at each instant of time is proportional to the product of its mass and its virtual velocity.[30]

Secondly one can consider the force of a body in motion as the sum of all the forces which have been present in the motion of a body i.e. as the sum of all the dead or

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[30] *Ibid.*, 190-192.

instantaneous forces which have accompanied a body during its motion. Since each of the instantaneous forces was as the product of the mass and the [virtual] velocity of a body, the sum of all the small instantaneous forces present in the motion of a body will be as the product of its mass and the sum of all the velocities accompanying its motion. But the sum of these velocities is as the space traversed. Thus this estimate of force is as the mass of the body multiplied by the space traversed. If p, π are the sums of the dead forces accompanying m, μ one has: $p:\pi :: me : \mu\epsilon$. Substituting p, π for $me:\mu\epsilon$ in formulas A, B, C, D we have:

$$p:\pi :: \begin{cases} ftt : \varphi\theta\theta \\ \varphi mmuu : f\mu\mu vv \\ fetv : \varphi\epsilon\theta u \\ mut : \mu v\theta \end{cases}$$

These dead forces, (p, π) however, considered as the sum of all the forces accompanying masses in motion are not the <u>force vives</u> of a body in the sense of Bernoulli. These forces do not exist at the same time in a body which moves but exist successivly. Bernoulli did not take the instantaneous forces accompanying a body for <u>force vive</u>.[31]

Finally one can consider the forces of bodies in motion such that they are capable of producing effects

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[31] *Ibid.*, 192-195.

and surmounting obstacles. These are as their masses multiplied by the squares of their velocities. When a body in motion surmounts for example a spring by compressing it, it finds as obstacles to its motion the number of elastic parts of the spring and their force. If the number of coils of the springs are represented by e, ϵ or the spaces they occupy, and their forces (roideurs) by f, φ, the obstacles to the bodies in motion of masses m, μ, are $fe : \varphi\epsilon$. One always has $fe:\varphi\epsilon::muu:\mu vv$ so the obstacles which can overcome bodies in motion are as the products of their masses and the squares of their velocities. That is why in estimating the forces of bodies in motion with respect to the obstacles they can overcome, one will have these forces as the products of their masses and the squares of their velocities. It is only force taken in this sense which should be taken as "force vive. This then is the meaning of Bernoulli's force vive.[32]

Jacques Eugène de Louville's (1671-1732) essay On the Theory of Varying Motions, which are Continually Accelerated or Retarded, With the Method of Estimating the Force of Bodies in Motion,[33] attempted to apply the Cartesian esti-

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[32] Ibid., 195, 196.

[33] M. le Chevalier Jacques Eugène de Louville, "Sur la théorie des mouvements variés, c'est à dire, qui sont continuellement accélérés ou continuellement retardés; avec la mannière d' estimer la force des corps en mouvement," Histoire de l'academie royale des science (1729) M. 154-184.

mate of force to Bernoulli's problem of the *vis viva* acquired by bodies under the action of expanding springs. He showed that other concepts of force which he termed instantaneous, actual, and virtual force could be used to explain this action. However, in so doing he considered himself to have refuted the validity of the *vis viva* concept.

Because a body at rest cannot move another, nor move itself, it is necessary for something to happen that changes rest into motion. That something is moving force (*force motrice*) or simply force. Four things are necessary to produce motion - force, (f) a body, (m) space, (e) and time (t). These four produce a fifth, velocity (u). This has a certain relation to force and mass, and again to space and time.[34]

The relationship of the velocity to the force and mass is expressed as $u = \frac{f}{m}$ which gives $f = mu$. This is disputed by clever geometers but, says Louville, he hopes in this memoir to establish it in so firm a manner that no doubt will remain.

The velocity is proportional to the space divided by the time or $u = \frac{e}{t}$. Thus if $f = mu$ and u in turn is to equal e/t, then $f = \frac{me}{t}$ and not $f = me$ as some geometers

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[34] *Ibid.*, 154-156.

have put forth. This latter is true only when the time during which the spaces are traversed is the same. Then the space is as the velocity and represents it. Hence $\underline{f} = \underline{me}$ for this single case.[35]

The equation $\underline{f} = \underline{mu}$ shows that when the velocities of two bodies are in a ratio reciprocal to their masses, these bodies have equal forces. Also when the forces of two bodies are equal, their velocities are in a ratio reciprocal to their masses.

Following the hypothesis of Galileo, a body under the action of gravity whether it rises or falls receives in equal times an equal number of impulsions of velocity which push it downward imprinting on it accelerations. The sum of these accelerations is as the time, so the sum of the impulsions increases as the number of accelerations and consequently as the time. The space in free fall increases twice in the ratio of the time while the velocity increases only once. Hence the space increases in the double ratio [square] of the time [$e = \frac{1}{2}gt^2$] and the velocity increases only in the simple ratio of the time, [$u = gt$].[36]

Expressing this mathematically, when the velocity of a moving body increases or diminishes continually in the

[35] *Ibid.*, 156.

[36] *Ibid.*, 156.

ratio of the time, one has *u* = *t*. Putting the expression *e/t* in place of *u*, [in u=gt] one has *e/t* = *t* or *e* = *tt*. [Acutally $e=\frac{gtt}{2}$]. Hence the *spaces* traversed are as the squares of the times.[37]

As to the *force* that the falling body acquires by the action of gravity it is evident that this increases, as does the velocity, only by the number of impulsions of the fluid which is the cause of gravity. It is clear that a body which receives an impulsion which imprints a certain degree of force at the same time that it imprints a degree of velocity, never increases the force or the velocity of a body during the intervals of time between the impulsions. It is only the space traversed which increases during these intervals. Thus the force is never increased in the double ratio [square] of the time as is the space, but in the simple ratio of the time or as the velocity.[38] [This is the Cartesian definition of force, *mv*.]

Louville defines "instantaneous force or velocity" (*Vitesse instantanée*) and "actual force or velocity" (*vitesse actuelle*) to be the same things designated by Leibniz as dead force and living force. (Actually as will be seen Louville's and Leibniz's "forces" here are entirely

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[37] *Ibid.*, 158, 159. Louville's reasoning is the basis for the relationship: $\underline{mv}=\frac{1}{2}\underline{mgt^2}$. See this dissertation introduction, p. 9.

[38] *Ibid.*, 158, 159.

different concepts). The latter has not been sufficiently well clarified, he says, and will be in the essay. The force of each impulsion [F_i] communicated only in an instant is "instantaneous force". "Actual force" is the product of the force of each impulsion [F_i] by the number [sum] of impulsions the moving body receives in equal times, [$\int_{t_1}^{t_2} F_i dt$]. The general rule for all hypotheses of acceleration or resistance is that the effect of the acceleration or resistance is always in a ratio composed of the size of the impulsions by their number, or as the product of these two quantities. In this essay Louville is struggling to define the impulse of a force, that is, an impulsive force which varies with time. He does this mainly with respect to expanding springs. These forces are not constant; for a compressed spring the force starts at a maximum and decreases as the acceleration of the body starts at zero and increases to a maximum. The total number or sum of the elements of the instantaneous forces in equal units of time is the integral of the impulses, $\int_{t_1}^{t_2} \underline{Fdt}$, using modern notation. The total "force" which an expanding spring communicates to a body is Louville's "actual force". Louville considers this to be the correct measure of what Leibniz called living force. Actually he is defining a different concept: the impulse of the force which is equal to the momentum.

Louville then goes on to discuss the force of <u>elastic</u>

bodies in motion or of elastic springs, which can be impressed on other bodies, causing their acceleration.

A body has force only when it is in motion. An invisible material, for example a fluid in motion must therefore be the cause of the force of elasticity since when an elastic body resists compression no visible cause of this force is nearby. This material acts in the manner of fluids which do not act with their total mass on obstacles but hit these obstacles with repeated and successive impulsions. The obstacles yield to the effort of these impulses accelerating with velocities proportional to the velocity of the fluid and the number of impulses imprinted in equal times. That is, the effect of this acceleration is in a ratio composed of the velocity or the size of each impulsion and of the number of impulsions.[39]

The same idea is applicable to a weight placed on a horizontal plane. If the plane is solid enough to resist the weight of the body, the body stays at rest not having enough force to communicate to the plane supporting it. Properly speaking it is not this body at rest which presses the plane but an invisible fluid which continually hits it and of which the impulsions are all equal in force and of which their number is as the time.

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[39] *Ibid.*, 167.

When the obstacle which resists the effort of a compressed elastic body or the weight of a body is insurmountable, then regard must be paid only to the force of each impulsion. This is what is called "instantaneous force" (force instantanée) and Louville claims it to be what Leibniz and Bernoulli have called dead force. "Everyone agrees" that this force is as the quantity of motion or as the product of the mass by the "velocity" of each impulse. Here Louville should say the "virtual velocity"; dead force = $m\frac{dv}{dt}$ = mdv, for static forces; Bernoulli's measure of dv was given as Pdx. These impulses do not accumulate as they do in the case of a body which yields, so each is extinguished in the instant it acts. They perish on being born as Bernoulli says, and their effort never survives their action. If the obstacle has enough force to resist the first impulse it will resist the second and third etc. If, however, the obstacle yields to the first impulse, then all the impulses accumulate together, the first imprinting a small degree of velocity, the second adding a second degree, the third adding a third etc. The fluid thus acts on the body so as to accelerate it so that "the total effect of these accelerations in a given time till be as the product of the force of each impulsion [F_i] by the number of impulsions imprinted in the duration of its action" [$\int_{t_1}^{t_2} F_i dt$].[40]

[40] Ibid., 169-171.

Louville seems to be saying that his concept of instantaneous force $[F_i]$, or the force of each impulsion, is the same as Bernoulli's pressure, *p*, of the compressed spring. ~~Actually these are not identical.~~ Bernoulli's concept leads to a space-dependent function: $\int_{x_1}^{x_2} \underline{pdx} = \underline{mv^2}$; Louville's leads to a time-dependent function $\int_{t_1}^{t_2} \underline{Fdt} = \Delta \underline{mv}$.

Thus far Louville agrees, he says, with Leibniz and Bernoulli and also agrees that three kinds of force must be distinguished as will be discussed subsequently. However, he does not agree with what they have called living force (*force vive*) because they have called by the same name two different kinds of force. One cannot say that all living force is as the product of the mass of a body by the square of its velocity.[41]

Going on, Louville distinguishes two kinds of force, actual and virtual, in the acceleration caused by the expansion of elastic bodies, or springs.

While the impulses of the fluid which causes elasticity act continually they do not act continuously, that is, without intermission. One has no "clear and distinct idea" of a continuous impulse. An impulse acts only by repeated and successive shocks, with small intervals of time between them. To determine the effect of an acceleration in a given time it is necessary to know besides the force of each impulse, $[F_i]$ the frequency or number

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[41] *Ibid.*, 171.

[i.e. the sum of the impulses] that the fluid can imprint in the given time. The product of these two quantities is called the actual force (force actuelle) of the fluid [$F_i$dt]. "Thus to produce a certain acceleration, this force [for elastic bodies] depends not only on the size of each impulse but also on their frequency. Knowing what force will be produced in an infinitely small amount of time [dt], one can find the force over an extended time [t]."[42] Louville means that the instantaneous forces of elastic bodies are not constant, but vary with time.

Finally there is a third kind of force which must be considered, "virtual force" (force virtuelle). This pertains chiefly to the accelerations caused by groups of similar elastic springs but composed of a different number of parts. Each of these groups can produce an acceleration during its total expansion. Those springs composed of a great number of parts equally compressed, will follow the moving body which it accelerates over a longer time and path than a spring of a lesser number of parts. But since the frequency of the impulsions will be less for the spring with the larger number than for the similar one with the small number, the bodies accelerated will receive the same velocity.

The virtual force or virtual velocity of each group

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[42] Ibid., 171, 172.

of springs is equal in all groups composed of equal similar springs equally compressed, differing only in the number of elastic parts of which they are composed. The acceleration which they produce lasts only as long as the body is in contact with the spring and ceases when it leaves the spring. It is not clear by Louville refers to the forces of these similar springs as "virtual."

These two kinds of force, *force actuelle* and *force virtuelle*, says Louville, have been confused by Leibniz and others by calling both living force.[43] Again Louville is referring to a different concept (impulse) and not Leibniz's living force, mv^2. Thus here the problem is again a misunderstanding over words.

Louville applies the distinction among the three kinds of force to Bernoulli's argument for living force drawn from the expansion of similar springs.[44] What Bernoulli has called dead force or pressure in this proof and what he, Louville, had defined as instantaneous force, or impulsion is not the quantity which produces an effect proportional to the time of its action. It is not sufficient to know the size of this force to determine the acceleration in a given time. It is necessary also to know the frequency of the impulsions.

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[43] *Ibid.*, 172, 173, 177.

[44] See Bernoulli, *op. cit.*, Chapter VII.

It is Louville's <u>force actuelle</u> which is indeed composed of the ratio of the magnitude of the force of each impulse and of the number of impressions in equal times.[45] But again behind Bernoulli's argument is the relationship $\frac{1}{2}mv^2 = F \cdot ds$; Louville is developing the equivalency ${}_{t_1}\int^{t_2} F dt = {}_{v_1}\int^{v_2} m dv = \Delta mv$.

Secondly, the <u>force lost by the spring</u> in accelerating the moving body, <u>is equal</u> to the <u>force gained by the body</u> after leaving its contact with the spring. The force of the spring is entirely exhausted and is transferred to the body in which it is taken up by the accumulation of the small successively produced degrees of force. This force which is transferred is what Louville has called the <u>virtual force</u> of the spring.[46]

In Bernoulli's example each of the two springs have lost the same amount of force in traversing their spaces and the two moving bodies have gained as much force in traversing the same spaces. Each have the same amount of force when they abandon their springs. Since the masses are equal, it follows that their velocities will also be equal. Thus the velocities of bodies in motion are as

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[45] <u>Ibid</u>., 176.

[46] <u>Ibid</u>., 178.

their forces.[47] Here Louville relates the impulse of the force to the momentum gained by the accelerating body, *mv*.

Thus although Louville has succeeded in showing that the moving bodies have acquired momentum (*mv*) from the expansion of the compressed springs, he has not successfully demonstrated that they do not also acquire kinetic energy. This is because he has confused Leibniz and Bernoulli's living force, mv^2, with the concept he has defined, impulse, which is equivalent to the change in a body's momentum.

The final paper to be discussed in this series of articles appearing in the journals of the French Academy in the 1720's is that of Jean Jacques de Mairan, secretary of the Academy. His "Dissertation on the Estimation and Measure of the Moving Forces of Bodies",[48] is primarily an unsuccessful attempt to reduce cases of accelerated and retarded motion where the *vis viva* principle appears to hold, to cases of uniform motion where momentum, *mv*, is valid.

Mairan's paper was later hailed by the Academy as having settled the issue perhaps because he was the Academy's

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[47] *Ibid.*, 178, 179.

[48] Jean Jacques de Mairan, "Dissertation sur l'estimation et la mesure des forces motrices des corps," Histoire de *l'academie royale des sciences* (1728), M. 1-49.

secretary. It was this paper, however, which touched off a renewed debate in the 1740's when it was attacked by Madame du Châtelet and as a consequence reprinted.

Mairan presupposed that nature behaved as in most of her phenomena, in a perfectly uniform manner with regard to the forces of moving bodies.

He argues for the importance of using uniform motion in measuring force. If a force is applied to a body its effect is movement. If the force does not impede movement it will produce it. Movement can be uniform or non-uniform which in turn can be accelerated or retarded. In uniform motion the effect is that of equal spaces traversed in equal times. Uniform motion or velocity itself is the space divided by the time. Quantity of motion is measured by the mass times its velocity i.e. by uniform motion. If two bodies <u>A</u> and <u>B</u> of the same mass move uniformly with the same force and with the same velocity but one moves for one hour and the other for two hours they have two different quantities of motion, in the ratio of 1 to 2. Those bodies whose movement is not uniform do not represent nature as she is.

In the collision of infinitely <u>hard</u> and inflexible bodies the evaluation of force as producing uniform motion is unchanged because the collision and communication of motion are instantaneous and do not suspend the uniformity of the motion. Only the velocity after collision is changed, the same force being scattered afterwards on the other bodies encountered and with which a body continues to move uniformly but with a lesser velocity in the inverse ratio of the masses.

In most bodies there is an "elastic virtue" which acts by compression and restitution of the parts of the body displaced in collision.

The only difference between elastic and hard bodies is that in the former the communication of force between them is successive and in the latter it is instantaneous.

In summarizing Leibniz's argument, Mairan shows that dead force such as that in a body placed on a table making a continual effort to descend is measured by <u>mv</u>. Here again the confusion between actual velocity <u>v</u>, and virtual velocity, <u>dv</u>, appears. Living force, measured by $\underline{mv}^2$ is calculated from its effect or the height to which a body of a certain velocity and mass can rise if thrown upward.

But, says Mairan, the great principle must not be forgotten which says that proportion implies a common measure. That common measure is the time. The time or equal times must be taken in determining the common measure of the forces to be compared. A body having two times more velocity than another has a double and not a quadruple effect, a double traversed space and a double displacement in equal times. Thus a moving force has only double and not quadruple effect, it is as the simple velocity and not as the square of the velocity.[49]

In demonstrating the measure of force as <u>mv</u> Mairan uses equal times as the common measure for comparing forces and employs a technique of Louville for reducing accelerated or retarded motion to uniform motion.[50] This means that force will be measured not by the spaces traversed in non-uniform motion, nor by the obstacles overcome, nor by the parts of matter displaced, nor by the distortion of elastic bodies, but by the spaces <u>not</u> traversed which would be traversed if the motion were uniform, by the obstacles <u>not</u> overcome, the parts of matter <u>not</u> displaced, the elasticity <u>not</u> distorted.[51]

Accelerated motion is reduced to uniform by the following technique.[52] Two equal bodies <u>A</u> and <u>B</u>, <u>A</u> having a velocity of 2 and <u>B</u> a velocity of 1 ascend along two paths, (see accompanying figure, Mairan (1728) p. 30)

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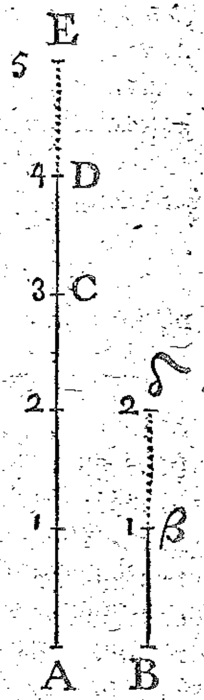
[49]*Ibid.*, secs. 1-15.

[50]See Annon., "Sur la force des corps en Mouvement" *Histoire de l'Academie royale des sciences*, (1721) H 81-85.

[51]Mairan, *op. cit.*, 38, 40, 41.

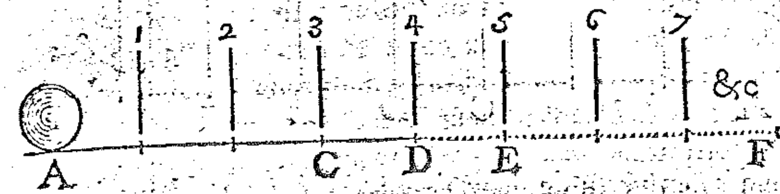
[52]*Ibid.*, sec. 39.

avoit 2 degrés de Force, & 2 degrés de vîtesse, il lui en reste encore 1, & il se trouve par là en C, & à la fin du premier temps, dans le cas où se trouvoit le corps B au commencement de ce premier temps. Il a donc tout ce qu'il faut pour parcourir encore 2 toises CE, en un second temps semblable au premier, si aucune impulsion contraire ne s'y oppose. Mais les impulsions contraires de la Pesanteur vont s'y opposer, & de la même façon precisément qu'elles se sont opposées au Mouvement du corps B. Donc le corps A ne parcourra pendant ce 2me temps, que la toise CD, ayant pour ainsi dire, reculé de l'autre toise, ED, en vertu du retardement, ou des impulsions contraires à sa Force Motrice; après quoi il s'arrestera en D, ou ne montera plus, comme le corps B en β. De sorte qu'il n'aura parcouru en tout dans les 2 temps de son Mouvement, que 4 toises. Ce sont ces espaces $\beta\delta$, CD, dans le premier instant, & DE, dans le second, & ainsi de suite, que j'appelle *non parcourus*. Ils sont non parcourus, relativement à la Force Motrice des corps A, & B, & à leur direction donnée de B vers δ, & de A vers E, à laquelle seule on fait attention; quoique en un sens, ils soient très réellement parcourus en valeur, en direction contraire, & par l'effet d'une autre Force Motrice opposée à la premiere, qui s'y mêle, & qui la modifie continuellement, comme feroit le Mouvement contraire d'un plan sur lequel le Mobile seroit porté.



40. Ce qui est dit ici des espaces non parcourus n'a pas moins lieu à l'égard de tous les autres effets du Mouvement, & du choc, comme il a été remarqué ci-dessus (N.° 27.) par rapport aux espaces parcourus. Et nous dirons de même, 1.° *Que ce ne sont pas les parties de matiere deplacées, ni les ressorts bandés ou applatis, qui donnent l'Estimation & la mesure de la Force Motrice, mais les parties de matiere* non déplacées, *les ressorts* non bandés *ou* non applatis, *& qui l'auroient été, si la Force Motrice se fut toûjours soûtenuë & n'eût point souffert de diminution.* 2.° *Que ces parties de matiere* non déplacées *sont en raison, &c.* Comme N.° 38.

41. Pour en donner un exemple, soient des impulsions, des obstacles, ou des résistances quelconques uniformément repetées, & placées sur le chemin AF, du Mobile A, telles par exemple, que les particules de matiere 1. 2. 3. 4. &c.



ou des lames de ressort à déplacer, à abbatre, à soûlever, ou à bander. Il est évident que si le Mobile, avec un degré de vîtesse, & de Force, peut en soûlever 2 en un instant, par un Mouvement uniforme, c'est-à-dire, en conservant, ou en reprenant toûjours toute sa Force, & toute sa vîtesse, après avoir soûlevé la premiere; & qu'au contraire, il n'en puisse soûlever qu'une par un Mouvement retardé, toute sa Force, & toute sa vîtesse s'estant consumée à soûlever, ou à bander la premiere, il est, dis-je, évident par tout ce qui a été dit ci-dessus (N.° 15, 28) que le Mobile A ayant 2 degrés de Force, & autant de vîtesse, soûleveroit, ou banderoit 4 de ces lames de ressort dans un instant par un Mouvement uniforme. Mais il perd dans cet instant, & en bandant les premiers ressorts, un degré de sa Force, & de sa vîtesse; & un degré de Force & de vîtesse perduë donne, par hypothese (N.° 27) une lame de moins soûlevée, ou bandée; donc il n'en bandera que 3 au premier instant, sçavoir 1, 2, 3, & il s'en faudra la lame 4, & l'espace CD, qu'il ne fasse ce qu'il auroit fait s'il n'eut rien perdu. Cependant, comme il lui reste encore un degré de Force, & de vîtesse, qui lui feroit soûlever deux lames 4, 5, & parcourir le chemin CDE.

First imagine the ascent to be possible with uniform motion. Let _B_ travel 2 _toises_ in the first instant; _A_, having double the velocity, will then travel 4 in the first instant. However, under motion retarded by gravity, _B_ having a velocity of 1 will rise only 1 toise since, as Galileo showed, in equal times the spaces traversed by a body traveling uniformly is twice that traversed by a uniformly accelerated body starting from rest and reaching a final velocity equal to the average velocity of the first. Since its velocity is only 1, it will travel only one unit of space in its motion retarded by the force of gravity. Thus the distance _not_ traveled, which would be if the motion were uniform, is 2 - 1 = 1. Body _A_ having a velocity of 2 will rise in retarded motion to a total height of 4, 3 units of which will be traversed in the first instant and one in the second instant. Therefore, for body _A_ the distance _not_ traveled in the first instant is the uniform motion of _A_ minus its retarded motion or 4 - 3 = 1. In the second instant of retarded motion, _A_ rises 1 unit, but it would have traveled 2 under uniform motion. The distance not traveled in the second instant is again 1.[53] Thus the spaces not traversed by _A_, having a double velocity, equal 2. These are double those not traversed by _B_, i.e. 1. The

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[53]De Mairan, "Dissertation," 1728, 539. See also Dugas, _A History of Mechanics_, _op. cit._, p. 237.

spaces not traversed in each instant represent the force lost or consumed in each instant, or the effort of the contrary force which destroys or consumes it. But the sum of all the lost forces or of the contrary forces is equal to the total force of the body.(§ 43).

The advocates of living force would say that of these two equal bodies, A and B, the force of A ($v = 2$) is 4 since it rises a distance of 4 while that of B ($v = 1$) is 1. Mairan can now retort that the force of A is 2 and that of B is 1. In spite of its ingenuity, the weakness of his hypothesis is apparent, as Madame du Châtelet was later to point out.[54] For the actual situation is that of retarded motion, not the hypothetical one of uniform motion.

For the case of a body traveling with uniform motion in a horizontal plane meeting a series of obstacles, Mairan argues as follows. Imagine 100 equal elastic balls A, B, C, D etc. ranging along the horizontal line HL, moving one after another by virtue of a single force and imagine a single motion imprinted on the first ball A in the direction HL. It is not possible to measure the force applied to ball A by the product of its velocity and the 100 masses sharing in the motion because they share successively only and a single ball moves in each instant under consideration.

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[54]See this dissertation, Ch. IX, p. 339.

The sum does not express the measure of the primitive cause, but rather the simple repetition or index of its duration in regard to the contrary causes which are able to destroy or arrest its action.[55]

De Mairan sets up the following argument:

> For example let there be impulsions, obstacles, or any resistances whatever repeated and placed on the path AF of moving body A. /These can be/, for example particles of matter 1,2,3,4, etc., or elastic strips (*lames de ressort*) to be displaced, knocked down, lifted, or bent.[56]

These elastic bands which represent the impulsions of gravity and which were later represented as elastic springs by Madame du Châtelet, are diagramed by Mairan in the preceding diagram (see diagram p. 316). These little springs or flexible strips are utilized by most of the later authors in their arguments. Their use in this controversy seems to stem from Bernoulli's spring demonstration. From the above mentioned diagram, Mairan argues:

If a body with one degree of velocity and of force can lift 2 elastic strips in one instant by a uniform motion, it will by a motion retarded in collision with the first strip, consume all its force and velocity in lifting it. The body A however having 2 degrees of force

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[55] *Ibid.*, sec. 3..

and velocity can lift or bend 4 of the elastic strips in one instant by a uniform motion. But if in this first instant in bending the first band it loses one degree of its force or velocity, it will bend a total of only three bands in this first instant. It will not hit strip 4, or space CD, which it would have, had it lost no velocity. If it then continues to move for a second instant now having only 1 degree of velocity it will lift strips 4 and 5 in traversing path CDE if its motion is considered uniform. But because its motion is retarded by hitting strip 4 it will stop there having lost all its motion. There will be a total of 4 elastic bands lifted, and 2 not lifted or 2 degrees of force resulting from 2 degrees of velocity acting over a time of 2 instants.

If instead of supposing 2 degrees of velocity and 2 instants, one supposes the velocity to be 3 or 4 the body will traverse 6 or 8 spaces, displacing 6 or 8 strips by uniform motion. But by retarded motion it displaces 6-1, or 8-1 in the first instant etc. The total space traversed is 3 or 4 and the total not traversed is also 3 or 4. Therefore the portions of matter not displaced, the elastics not lifted, or bent, the objects not flattened, and in general the obstacles not surmounted which would be under uniform motion are proportional to the forces or simple velocities.[57] The spaces not traversed in each instant

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[57] *Ibid.*, sec. 41.

represent the force lost and consumed in each instant, or in other words they represent the effect of a contrary force exercising itself against the original force. But the sum of all the lost forces or of all the contrary efforts is equal to the total force of the body.

Thus Mairan by changing retarded motion to a case of uniform motion, changed a vis viva problem into a momentum problem. This technique inspired Madame du Châtelet in 1740 to write a long criticism of Mairan in defense of the Leibnizian position in dynamics.[58]

Conclusion

In this chapter it was seen that another important mechanical problem provided the stimulus for continued debate over the measure of force, the problem of the acceleration of bodies due to expanding elastic springs. The problem originated with Bernoulli's "Discourse on the Laws of the Communication of Motion" (1727), twice submitted

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[58] Three papers submitted for the first volume of the Academy of Petersbourg which discuss living force have not been analyzed in this dissertation. One argument from one of these papers, that of Hermann, is mentioned by Madame du Châtelet. However, arguments from these papers are periferal to the problems under debate by most authors discussed in this dissertation. The papers are:

Georg Bernhard Bulfinger, "Demonstrationes mechanicae de viribus corpori moto insitis et illarum mensura," Commentarii Academiae Scientiarum Imperialis Petropolitanae, 1 (1728) 45-120. Jacob Hermann, "De Mensura virum corporum," Ibid., 1-42. Christian Wolff, "Principia Dynamica," Ibid., 217-238.

for contests sponsored by the French Academy. Bernoulli had defended the Leibnizian measure of force $\underline{mv}^2$, its conservation in elastic impact, and the Leibnizian philosophy which denied the existence of hard bodies. His argument was based on a comparison of the forces ($\underline{mv}^2$) received by two bodies under the action of two similar expanding springs.

Three papers resulted in reaction to Bernoulli's "Discourse": Camus (1728), Louville (1729), and Mairan (1728). Camus related the force of compressed springs to the *vis viva* acquired by bodies as the springs expanded. He compared the resistance of a compressed spring to the resistance due to gravity which an ascending body finds at its summit. The velocity with which a body can ascend a certain curve will be the velocity with which it can compress a spring of the same resistance. The laws of accelerated motion under the action of gravity will then also be the laws of accelerated motion of elastic springs, i.e. as the square of the body's velocity.

Louville (1729) on the other hand, demonstrated that a body accelerated by an expanding spring would acquire a momentum, *mv*. He reached this conclusion by describingthe impulse of the force of an expanding spring. He implied that the total number, or sum of the elements of the instantaneous forces in equal units of time is the integral of the impulses, expressed by the modern equation: $\int_{t_1}^{t_2} \underline{Fdt}$.

Louville then said that the force lost by the spring in accelerating the moving body was equal to the force gained by the body as it left the spring. He showed that the body would acquire a velocity which depended on the impulse.

Mairan attempted to refute Leibniz by reducing cases of accelerated or retarded motion to cases of uniform motion. He argued that force should be measured not by the spaces traversed in non-uniform motion, nor by the distortion of elastic bodies, but by the spaces _not_ traversed which would be if the motion were uniform or by the elastic parts of a body which were _not_ distorted. This technique resulted in _mv_ as a measure of force but depended upon the substitution of a hypothetical situation for the real one. He applied his method to problems in which moving bodies collided with a series of vertically situated elastic strips (_lames de ressort_) which were lifted as the body moved past them. The number of elastic bands _not_ raised as the body moves under uniform motion compared to those raised under retarded motion is as the simple velocity. Again Mairan concluded from this that _mv_ is the measure of force. This technique was challenged by Madame du Châtelet in 1740 and as result the controversy was reopened. The re-examination culminated in the insights of Boscovich and d'Alembert that both measures had their own validity. These arguments will be examined in Chapter IX.