CHAPTER IX

Re-Examination of Living Force in the 1740's

In 1740 the controversy over living forces was re-opened by Madame du Châtelet who in an appendix to her Institutions de physique challenged Mairan's paper.[1] Mairan and du Châtelet exchanged criticisms, the publication of which inspired comments from other writers, Voltaire (1741) and Abbé Diedier (1743). D'Alembert's first edition of the Traité de dynamique (1743) discussed the controversy but did not resolve the problem. Only after additional contributions from Boscovich (1745), did d'Alembert (1758) give a full statement of the difficulties dividing the two camps.

Du Châtelet's book which appeared annonymously in Paris in 1740, although some title pages say London, 1741, was meant as a text book for her son's use. In part an attempt to popularize Leibniz's views, it was successful in creating immediate interest and excitement. Du Châtelet, the mistress of Voltaire, had in her first years of association with him beginning in 1735, been primarily a student of Newtonian

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[1] Gabrielle Emelie du Châtelet, Institutions de physique, Paris, 1740, Ch. 20,21.

physics. In 1738, however, she read Bernoulli's "Discours sur les lois de la communication du movement" and was converted to the Leibnizian position, at least, in dynamics. Then in March 1739, Samuel Koenig, was brought to Cirey by Maupertuis as a tutor for her and Voltaire in mathematics. By way of Koenig she came under the influence of Leibniz's philosophical views through their expression in the ideas of Christian Wolff. As a result of Koenig's teachings she revised her manuscript of the _Institutions de physique_ so that although it was Newtonian in its basic principles, it followed Leibniz on the subject of dynamics.[2]

Du Châtelet begins with Leibniz's distinction between dead and living force. Living force can arise from dead force when a body is continually subject to a series of infinitely small forces or pressures /_pressions_/. If a body yields to these dead forces, it conserves them and acquires a force which is the sum of all these accumulated pressures.[3]

Du Châtelet gave two examples of the relationship of dead to living force. The first example of dead force was that of elasticity. She pictured a set of three similar sections of elastic springs (_ressorts_)"equally strong and equally tense" (See du Châtelet's figure 73, p. 327.

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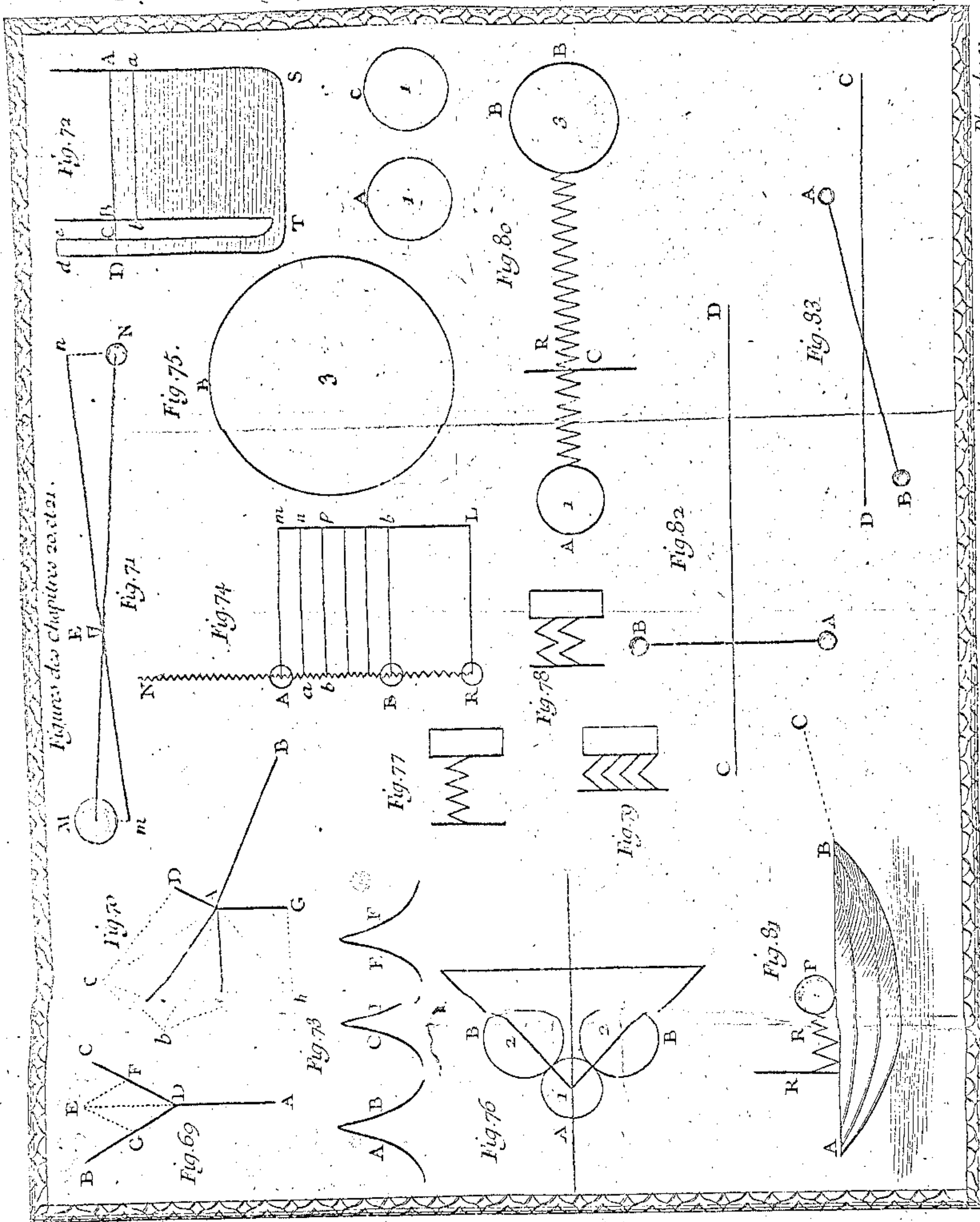
[2] W. H. Barber, _Leibniz in France from Arnauld to Voltaire. A Study in French Reactions to Leibnizianism, 1670-1760,_ Oxford, 1955, 135-140, 182-186.

[3] _Ibid._, sec., 567, p. 420.

If a body receives the force held in one of these elastic springs, a second body receiving the force held in two similar elastic springs equal to the first will acquire two times more force. A body receiving the force of 3 equal and similar springs will acquire three times the force. (See du Châtelet's figure, 73, p. 327.)

The second example is analogous: the force of gravity. Gravity presses uniformly on heavy bodies at each instant and at all points of their fall. Gravity can be considered as an infinite elastic spring NR pressing equally on body A in the space AB and acting at all points between A and B. (See du Châtelet's figure 74, p. 327). If one expresses the pressure on a body at A by the line Am, that which it receives in the second moment by the line an, the following pressure by line bp etc. up to the final position of body A at B, one finds that the sum of the rectangle formed by the pressures Am, an, bp etc. is the rectangle Ab. The living force acquired at B should be represented by this rectangle since it is composed of the sum of all the pressures received during the time the body moves from A to B. The living force of body A arriving at point B will be to that of body R descending from A to R as the rectangle Ab is to the rectangle AL. This is the same as the ratio AB to AR since rectangles of the same height are in the same ratio as their bases. The forces that the bodies have received at A and R are as the lines AB and AR since the living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forces are as the number of equal elastic springs communicating by expansion their forces to the bodies in motion. Since in a double space there are two times as many elastic coils as in a single space, the number of coils are in the ratio of the spaces AB to AR. Thus the living forces of the body descending by gravity are as the spaces AB to AR.

But these spaces are as the squares of the velocities, and thus the living forces of the bodies at B and R, are as the squares of their velocities.[4]

Du Châtelet replied to Mairan's assertion that time should be the common measure in comparing two forces, since the bodies in which double velocities produce quadruple effects do so in a double time, the forces being merely double when equal times are considered. She gave in response Leibniz's argument that the total force of a body should be measured from the time a body commences its movement until that instant in which it has exhausted all its force. "Time should enter into the consideration of force no more than into the measure of riches of a man, which are the same whether dispensed in a day, a year, or a hundred years."[5]

For force to be real and not merely a metaphysical

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[4]Du Châtelet, sec. 567, pp. 420-422.

[5]*Ibid.*, secs. 568, 569, pp. 423-424.

notion, a resistance is necessary by which its effects can be seen. If a body encounters other bodies which it sets in motion, or if it bends elastic springs or compresses or transports other masses, then the presence of the force is known and can be estimated by the quantity of the effects it produces.

The only occasion on which time should be considered is when a body moves for a long time under uniform motion. Then the only effect produced is the total space traversed and the effect depends on the length of the time of traversal. But a longer or shorter time will never alter the capacity of a body to remount to the height from which it fell or to compress a certain number of elastic springs. "If in a longer time a body could produce a greater effect as for example to rise to a height greater than that from which it fell, then perpetual motion would be possible... Thus the force destroyed is always equal to the effect it produces whatever the time."[6] To overcome a resistance of 100 it is always necessary to have 100 degrees of force no matter how long the time. Can a body which is stopped by hitting 3 elastic strips /lames de ressorts) in the first instant, be said to have the same force as a body which in the same time after hitting 3 obstacles, still has unconsumed force?[7]

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[6] Ibid., secs 570, 571, pp. 425, 426.

[7] Ibid., sec. 572, p. 426.

In refuting Mairan's supposition that force is measured by the spaces not traversed which would be under uniform motion, du Châtelet argues that two contradictory ideas are being used simultaneously. If a body exhausted a part of its force in closing (fermer) 3 elastic springs in the first second of its retarded motion, and only had enough force remaining to close one more in the next second, then it would have to take back some of its force if it could close 2 springs in the second second of uniform motion. The result would be, she says, that 2 + 2 = 6. A force can never in reality produce an effect greater than that which has destroyed it. It is contradictory to suppose at the same time that a force can remain the same and yet that it can produce a portion of the effects which consume it. Thus a force cannot be supposed at the same time to be uniform and also to have encountered a portion of the obstacles which would consume it.[8]

Du Châtelet then gives several arguments in support of the measure of living forces as mv^2. Of two travelers walking equally fast, one walks for one hour covering one league. The second traveler walks two leagues in two hours. The second traversed double the path of the first and,

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[8] Ibid., sec. 574, p. 432.

"everyone agrees", used double the force in so doing. Supposing now that a third traveler walks with a double velocity and covers the two leagues in only one hour. This third man used two times as much force as the second who took two hours to go the two leagues. It is now evident that this third traveler who used double the force of the second, who in turn used double the force of the first, must have used quadruple the force of the first, having walked double the space with double the velocity in the same time. Consequently the forces which the voyagers dispensed are as the squares of their velocities.[9]

The adversaries of living force, writes du Châtelet have always claimed that they would concede the argument if a case could be found in nature in which a double velocity produces a quadruple effect in the same time in which a simple velocity produces a simple effect. Such a case, she claimed to have found: A body A, suspended freely in the air, having velocity 2 and mass 1 hits at an angle of 60° (see diagrams, Fig. 76, p.327) two bodies B and B each having mass 2. The body A remains at rest after the collision, and the bodies B and B partake all its velocity between them, each moving off with velocity one. Each of these two bodies of mass 2 and velocity 1 have a force of 2. Thus body A with mass 1, velocity 2 communicated a force

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[9] Ibid., sec. 575.

of 4 in a single unit of time, precisely the case required by adversaries of living force. "Thus the objection drawn from the consideration of time is refuted."[10]

The "error" in an argument contrived by James Jurin was also exposed by Madame du Châtelet. Jurin had supposed a plane moving in a straight line with a velocity of 1. On this plane is a body of mass 1 acquiring its velocity from the moving plane and consequently having a force of 1. Now suppose that a spring capable of giving the body a velocity of one is fastened to the plane and in being released, pushes the body in the same direction as the plane. In so doing it communicates 1 degree of velocity and consequently one degree of force to the body. Now asks Jurin, what will be the total force of the body? The total force adds to 2, but the total velocity is also 2. Thus the force of a body is proportional to the mass multiplied by the simple velocity.

The error which du Châtelet finds in the above reasoning is this: Suppose for greater ease that in place of the plane of Jurin, a boat <u>AB</u> moves on a river in the direction <u>BC</u>, with velocity 1. (See Du Châtelet's figure 81, p. 327) Body <u>P</u> is transported on the boat acquiring thereby the same velocity as the boat. The elastic spring touching the ball is supported at the other end by an immobile support.

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[10] <u>Ibid</u>., sec. 581.

When released it pushes toward both directions, A and B, and communicates to the body P not a velocity of 1, but this velocity minus a second quantity which depends on the proportion between the mass of the boat, AB, and the mass of body P. The quantity of living force residing in the coiled spring, will, after its release, be found in the body and the boat taken together. Thus Jurin's case is founded on the false supposition that the elastic R will communicate to body P transported on a movable plane, the same force that it communicated to it when the spring was supported by an immovable obstacle at rest.[11]

A final argument cited by du Châtelet was one devised by Jacob Hermann in support of living forces. This argument prompted a reply and significant analysis by Mairan.

Ball A of mass 1 and velocity 2 collides first with ball B at rest having mass 3, and then with ball C of mass 1, also at rest. It gives a degree of its velocity to each, and having lost all its velocity is reduced to rest. The force of body B will be 3 on either the hypothesis of living forces or of quantity of motion. Likewise the force of body C is 1, making the total force after the collision equal to 4. On the basis of conservation of force, body A of mass 1 and velocity 2 must have had a force of 4 which

[11] Ibid., sec. 584.

is the square of the velocity multiplied by the mass.[12]

In response to the objections and arguments put forth in the Institutions de physique, de Mairan in 1741 wrote Madame du Châtelet a "Letter on the Question of Living Forces".[13] Hermann's example he discarded as due to the coincidence that 2 + 2 = 2 x 2. To avoid this equivocation, he assigned a velocity of 4, rather than 2, to body A. On colliding with body B, it will impart to it a velocity of 2 giving it a force (mv) of 3 x 2 = 6. A now is reflected in the contrary direction, communicates to C a velocity of -2 and is brought to rest. De Mairan now recognized the error in the sign of the velocity of mv. On the old basis of computation, the total quantity of motion, mv, after the collision is 6 + 2 = 8, whereas before the collision it is (4) (1) = 4.[14] Now instead, subtracting the negative quantity belonging to A after the collision from the positive quantity belonging to B, and considering the collision as taking place from the common center of gravity, 4 degrees of force residing jointly in A and B result from the collision.[15]

De Mairan points out, however, that on the doctrine

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[12] Ibid., sec. 577.

[13] De Mairan, Lettre, à Madame*** sur la question des forces vives en response aux objections, Paris, 1741, 1-37.

[14] Ibid., 20, 21.

[15] Ibid., pp. 25, 26, 28.

of living forces, the initial force of A would be (1) $(4)^2 = 16$. This he claims does not equal the force after the collision, whether it be 8 or 4.[16] Had he not made the mistake of substituting the total mv after the collision, and had instead calculated the total living force, he would have discovered that they add up to 16. In a response to his letter, to be discussed later, Madame du Châtelet, points out this fact. In Hermann's example also, if negative velocities are employed, both mv and mv^2 are conserved.

In his letter written in answer to Madame du Châtelet, de Mairan states that he has allowed his 1728 essay to be republished in order that the public may judge whether the paralogism she has announced is real or whether the argument rests on solid ground. She has, he says, discovered a pretended mistake in calculation by interpreting him as saying that the same force necessary to lift 4 elastic strips can also lift six, as if he had said that 2 + 2 = 6.

Imagine that 2 bodies M and N which are caused to rise vertically by the same impulse, one (M) by a retarded motion and the other by a uniform motion or an assemblage of uniform motions such that each at each instant is equal to the velocity of the body M at the beginning of the

[16] *Ibid.*, p. 22.

corresponding instant of its retarded motion. It follows that while the body M traverses eg. 5 ~~to~~ toises in the 1st instant, 3 in the second, and 1 in the 3rd, that N will traverse 6 toises in the 1st, 4 in the second, and 2 in the 3rd, or 12 toises in all.

In reducing retarded motion to uniform, one is simply following Galileo on the theory of movement and using the inverse supposition to his.

The three toises more, traversed by body N and not by M (i.e. 12-9 = 3) are due to the retardations of the primitive force of body M by gravity. Thus the primitive force of body M is as the simple velocity and not as the square of the velocity.

Madame du Châtelet, he claims has misrepresented his meaning by pretending to quote him but not including certain words, such as that the body "has a uniform motion at each instant."[17]

Du Chatelet's "undiscoverable" case of one ball, mass 1, hitting two others of mass 2 simultaneously, such that both move away at angles of 60°, thus producing a quedruple effect in a single unit of time, de Mairan attributes to the greater number of givens necessary to produce the effect. That is, the first ball must now collide with two balls to give the required velocity and moreover these

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[17] *Ibid.*, 1 - 16.

must separate at a certain angle. Moreover the effects in this example are due to the decomposition of forces in general and concludes nothing in favor of living forces. Three, four, or one hundred forces held in equilibrium or as dead forces in the same space and having the same value would produce nothing more.

In concluding his letter, de Mairan again states the great issue dividing the two camps. The proponents of living force say that time is nothing and velocity is all. "I say on the contrary, that the time is all and the velocity is nothing."[18]

In her turn Madame du Châtelet answered de Mairan, and his letter (February, 1741) together with her reply (March 1741) were bound with her Dissertation on the Nature and Propagation of Fire (published 1744).[19] She answers de Mairan's charge that she has not paid sufficient attention to crucial phrases in his argument. This is accomplished by comparing quotes from de Mairan's original paper and her paraphrase of it. She then reiterates the argument of the Institutions de physique that uniform motion and retarded

[18] Ibid., 31-37.

[19] Gabrielle Emelie du Châtelet, "Response de Madame la Marquise du Châtelet à la lettre que M. de Mairan, secretaire perpetual de l'academie royale des sciences, lui à écrite le 18. Février, 1741, sur la question des forces vives," Brussels, 1741 37pp, bound with Dissertation sur la nature et propagation du feu, Paris, 1744.

motion cannot be considered simultaneously. She claims that de Mairan says explicitly that body A encountering resistances placed on its path with a velocity of one and a force of one, consumes this in raising one elastic band /lames de ressort/ in the first instant, but then recovers all its force and all its velocity in order to raise 2 elastic strips in the first instant with uniform motion. This double situation is not at all possible.[20]

Similarly in the case where body A has a velocity of 2 de Mairan claims, says du Châtelet, that by unfiorm motion and a constant force 4 resistances will be raised in the first instant and 2 in the second. But by the hypothesis that the same body is moving under retarded motion it will raise 3 elastic bands with its 2 degrees of velocity in the first instant and one in the second instant. It is not permissable, argues Madame du Châtelet to suppose that at the same time 4 resistances can be raised and that they cannot be raised. If you say that the body A will raise 4 resistances in the first instant you cannot at the same time argue that part of its force will be consumed in raising them. If part of the force is consumed then the 2 additional resistances which de Mairan says would be raised in the second instant, will either not be raised at all or will be raised by virtue of new force from another agent. A body

[20] Ibid., 16. See de Mairan's diagram, this dissertation, Ch. VIII, p. 316.

cannot at one and the same time be considered as moving under uniform motion and under retarded motion. This is like supposing that at one and the same time 2 and 2 are 4 and 6.[21]

The case of bodies thrown upward with a certain velocity or of falling bodies remounting to the same height from which they fell is no more favorable than that discussed above of bodies encountering elastic resistances. Either the obstacles which the force of gravity presents to a rising body remain and hence the body with velocity 2 rises to height 4 or if they are removed and there is no force of gravity the body moves through the void, losing none of its force and none of its velocity. In general, the effects produced by uniform motion and retarded motion are different and cannot be compared. The effect of the first is only the space traversed, without obstacles encountered within it; that of the second consists in the displacement of these obstacles. In all those cases which are possible, the force of bodies should be evaluated by the obstacles which it is possible to overcome. It is not permitted to substitute for real parts actually overcome or consumed, imaginary parts that cannot be surmounted, without supposing at the same time, contradictions.[22]

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[21] *Ibid.*, 18, 19.

[22] *Ibid.*, 21.

Du Châtelet's analysis of de Mairan's attempt to reduce the _vis viva_ problems involving free fall and collision to problems of uniform motion, is exceedingly astute. Her statement that physically possible situations must be taken in the discussion is indeed the crucial issue.

The remainder of her letter to de Mairan is in defense of the "great geometer" Hermann against de Mairan's accusation that he had "confused the double of a quantity with its square."

Taking the balls, _A_, _B_, _C_, and giving to Ball _A_ a velocity 4, rather than 2, to remove the equivocation, it is obvious that in collision this will give to Ball _B_, mass 3, a velocity of 2. Thus ball _B_, has a force of 6, according to de Mairan. But says du Châtelet, according to her view the ball _B_ has a force of $3(2)^2 = 12$.[23]

Ball _A_ rebounding with velocity 2, mass 1, thereby has force 4. The total force of _A_ and _B_ after collision is 12 + 4 = 16 which is the same as that of body _A_ before collision $(1)(4)^2 = 16$. So instead of refuting Hermann the new case of de Mairan confirms him.

According to de Mairan's view, she says, Body _B_, mass 3, which obtained velocity 2 from body _A_, will have a "force" of 6. This is already more "force" than body _A_

[23] _Ibid._, 23.

had initially which was (1)(4) = 4. Furthermore body A, still has a "force" of (1) (2) = 2 left for itself. This compounds the absurdity. Du Châtelet questions de Mairan's use of the negative sign of the momentum of body A which would make the total momentum after the collision, i.e. 6 - 2 = 4, equal to that before the collision. The existence of force should not have to depend on which direction, whether to the left or to the right, the bodies are moving. When forces are calculated by the square of the velocity, this dependence on direction is irrelevant.[24]

The original case chosen by Hermann, she says, is neither fortuitous, nor particular, nor equivocal, as de Mairan claims, but completely general, for it is a case in which the adversaries of living force must agree with the proponents, depending as it does on the fact that unity is always equal to its square.[25]

Voltaire who joined the controversy in 1741 and who was the author of the popular French presentation of the principles of Newtonianism, was in every way opposed to the Leibnizian way of thinking. His skeptical, practical and empirical approach to science led him to impatience with any explanation of the world which went beyond the strictly material. The philosophy of Leibniz and that of

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[24] Ibid., 25, 26.

[25] Ibid., 27.

his follower Christian Wolff from whom Voltaire learned Leibnizian metaphysics, left him with little respect for Leibniz's views.

A French translation of one of Wolff's books appearing in 1736 was Voltaire's first introduction to Leibniz. His and Madame du Châtelet's association with Samuel Koenig in 1739 taught him more of Wolffian metaphysics and confirmed him in his opposition. For Voltaire the only scientific view of the world was the Newtonian. The empirical explanation common to English thought was far more natural to him.

Voltaire's loyalty to Madame du Châtelet caused him to restrain his attacks on Leibnizianism. However, he regretted her conversion and made fun of her enthusiasm for Leibniz. In spite of this he seems to have appreciated the merits of du Châtelet's Institutions de physique.

Voltaire's exposure to Lebiniz through Koenig lead to the publication of two books on Newton: Exposition des institutions physiques and the Metaphysique de Newton.[26]

An article by Voltaire, presented to the Academie des Sciences in 1741 and entitled "Doubts on the Measure of Motive Forces and their Nature," supported Mairan's

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[26] See W. H. Barber, Leibniz in France from Arnauld to Voltaire. A Study in French Reactions to Leibnizianism, 1670-1760. Oxford, 1955, 174-183, 191.

viewpoint and took issue with Hermann and Madame du Châtelet. In the first section on the measure of force, he writes that if any pressure /pression/ whatever can give in one unit of time, one unit of velocity to another body, this result will be one unit of force. If one unit of pressure, x, in one of time, t, can produce only one degree of force, then in the same time two pressures will produce two units of velocity or of force. "Thus in two units of time, one pressure will produce what two pressures could produce in one time." The result is "2 units of velocity or two units of force because 2x(t) = 2t(x)." Furthermore, "if of two equal bodies the first makes an effect double that of the second in an equal time, it will have double the velocity; if the effect is quadruple with double the velocity, then the time must be two units." If one wished the force to be the product of the square of the velocity by the mass, it would be necessary that a body with double the velocity should perform a quadruple action in the same time as an equal body having only a simple velocity. It would be necessary that the elastic body A equal to B and having velocity 2 should push a ball a distance of 4 in the same time as B with velocity 1 could push it a distance of 1.

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[27] Francois Voltaire, "Doutes sur la mesure des force motrices et sur leur nature, presentés à l'academie des sciences de Paris, en 1741," Oeuvres completes, Paris, 1819-1825 28:420-430. A summary of this essay appears in Histoire de l'Academie Royale des sciences (1741), Hist. 149-153.

This, however, is impossible.[28]

When accelerated or retarded motion is reduced to uniform motion these contradictions are resolved. Voltaire then summarizes de Mairan's argument in which a body thrown upward with two degrees of velocity needs two units of time to produce an effect quadruple that of an equal body thrown upward with one degree of velocity. But the spaces not traversed by these bodies represent the contrary force which destroys the original force of the bodies. The force destroyed in body A is only double that destroyed in body B in two units of time since the space not traversed by A, i.e. 2, is double that not traversed by B, i.e. 1.[29]

If force is only the product of a mass by its velocity, it is only the body itself acting or prepared to act with that velocity.

> Force is not, therefore, an internal principle /un principe interne/ a substance which animates bodies and is distinguished from bodies as some philosophers have maintained /i.e. Leibniz/.

"Force is nothing but the action of bodies in motion and does not exist primitively in simple beings called monads which these philosophers say are without extension and yet constitute extended matter.... They can no more produce moving force than zeros can form a number. If force is

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[28] Ibid., 420-421.

[29] Ibid., 422, 423.

only a property it is subject to variation as are all modes of matter. And if it is in the same ratio as the quantity of motion, is it not obvious that its quantity alters if the motion augments or diminishes?"[30]

In proving this point, Voltaire given an interesting incorrect example which strictly follows Descartes' concept of the quantity of motion. The quantity of motion is always increased when a small elastic body collides with a larger one at rest. For example the elastic body A of mass 20, in motion with velocity 11, (mv = 220) hits B at rest whose

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[30]Ibid., 428-429. See Leibniz, Correspondance with Arnauld, op. cit. 217, "...you will see what I mean M., when I say that a corporeal substance gives to itself its own motion, or, rather, whatever there is of reality in the motion at each moment, that is, the derivative force, of which it is the consequence; for every preceding state of a substance is consequence of its preceding state. It is true that a body which has no motion cannot give itself motion; but I hold there are no such bodies. (Also strictly speaking bodies are not pushed by others when there is contact but it is by their own motion or by the internal spring, which again is a motion of the internal parts. Every corporeal mass large or small, has already in it all the force that it will ever acquire, the contact with other bodies gives it only the determination, or, better, this determination takes place only at the time that the contact does)....p.221:"for I think rather that everything is full of animated bodies, and in my opinion there are incomparably more souls than M. Cordemoy has atoms. His atoms are finite in number while I hold that the number of souls, or at least of forms is wholly infinite, and that matter being divisible without end, no portion can be obtained so small that there are not in it animated bodies, or at least such as are endowed with primative entelechy, and (if you will permit me to use the word life so generally), with vital principle, that is to say, with corporeal substances, of all of which it may be said in general that they are alive."

mass is 200. (mv=0). A rebounds with a quantity of motion of 180, (mv = 180) and B goes forward with mv = 400. (mv=400). Thus A which originally had a force of 220, has produced a total force of 580. "But on the other hand, as everyone agrees, a great deal of motion is lost in the collision of inelastic bodies. Thus force in particular parts of matter increases and decreases."[31]

Thus the original invalid Cartesian rationale behind the quantity of motion existed for some thinkers unchanged and unchallenged almost 100 years after it was first put forth by Descartes. If Voltaire had taken into account the negative sign of the mv of ball A when rebounding, (i.e. - 180), then the total quantity of motion before and after the collision would be conserved /220 + 0 = -180 + 400/. Nor did Voltaire understand that momentum was conserved in inelastic collisions. The error in the sign of the velocity was still crucial.

Still another paper written by Abbé Deidier in 1741[32] reflects an attempt to reduce the vis viva problem to a momentum problem by taking as a standard for measurement the "force" a body would have provided if it were not moving under the restraining action of gravity.

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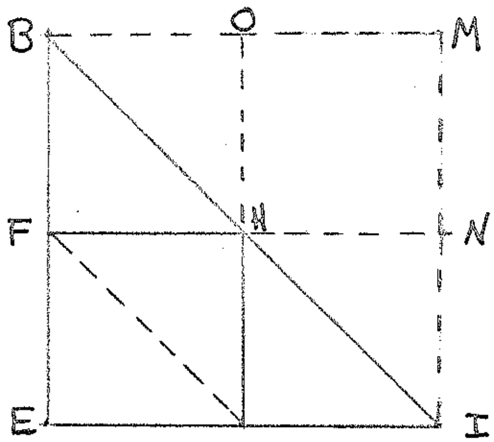
[31] Ibid., 428, 429.

[32] Abbé Deidier, Nouvelle refutation de l'hypotheses des force vives, Paris, 1741, 145 pp.

Deidier presents a "new refutation" of Leibniz's original 1686 argument establishing _vis viva_ as proportional to the spaces traversed in free fall. The difference in the forces of the two falling bodies, he says, comes not from the fact that the forces are in a different proportion from their simple velocities, but from the fact that the force which gravity removes from each of them in the same time is not proportional to their "primitive force". What does this mean? The force of body _A_, falling from a greater height, loses less in proportion than the force of body _B_ in an equal time. (See diagram, p.348). To demonstrate this, suppose that the first body descends during two equal units of time _BF_ and _FE_ and has traversed the space _BIE_, and that the second body during the first time, _BF_, has traversed the space _BFH_. According to Galileo's proportion the velocities are as the times _BE_, _BF_ or as 2 to 1. But if on remounting, these bodies are not subjected to the force of gravity, the first will traverse with a uniform motion in the two equal units of time _EF_, _FB_, a space equal to twice that of its descent, i.e. the space _EIMB_, the double of _BIE_. The second body during a time _FB_, equaling that of its descent, will traverse the space _FHOB_, or double the space _FHB_ traversed in its fall. Thus the first body in the time, _EF_ traverses only the space, _EINF_, which is only half the space, _EIMB_, which it traverses in a double time. Consequently the two bodies will traverse



Abbé Deidier, 1741

in an equal time the spaces EINF and FHBO which are in the ratio of their velocities of 2:1. But the masses multiplied by the spaces traversed in equal times are the measure of the forces. Thus the active forces (*force agissante*, or the force of a body in actual motion, as distinct from *force vive*, or living force which is proportional to $\underline{v}^2$) of the two equal bodies considered in equal times are as 2:1, or as the velocities when not under the action of gravity. What effect then does gravity have on them? In the first unit of time it removes a degree of velocity from the first body, and in the same time it removes a degree of velocity from the second body. This means that the second body, from which gravity has removed all its velocity, loses all its force. But this first, from which gravity removes only half of its velocity, loses only half of its force. Consequently its force lasts longer, not because its force or velocity is in a different ratio than 2:1 with the second body, but because less force is removed by gravity from the first in the same time than from the second. The spaces EIHF and FHB which are traversed in ascent in the same time are as 3:1, because the velocities removed from each by gravity are not proportional, the first body losing less in proportion to the second. It is evident then that this first body should traverse in the same time a space which is more than double that which the second traverses. This, however, does not diminish the primitive strength of the forces.[33]

Thus what is important in the measurement of force is the primitive force of a body not subject to any hindering forces such as that of gravity. The measure of forces by mv^2 is not legitimate not only because the fact that the times are different is neglected, but because the velocity removed in retarded motion in equal times are not proportional to the forces of the bodies.[34]

This argument of Deidier is similar to that of Mairan, the main difference being that Deidier's is given in terms of primitive force, whereas Mairan spoke of the uniform motion of a body.

Like many of the other champions of *mv*, Deidier presented a refutation of Bernoulli's *vis viva* proof depending on the nonuniform expansion of the elastic springs over a period of time.

In beginning that refutation he first calculates the times during which the two bodies are in motion. (See Bernoulli's, Figure 7, Ch. VIII, p.282). Let t be the time for ball P and T the time for ball L. Then by proportion, and by the general laws of mechanics:

$$t:T :: \int dx \frac{1}{\sqrt{\int pdx}} : n\int dx \frac{1}{\sqrt{n\int pdx}} :: \int dx \sqrt{n\int pdx} : n\int dx\sqrt{\int pdx}$$

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[33] *Ibid.*, 26-30.

[34] *Ibid.*, 34.

But $\int dx$ is the integral of DH and $n\int dx$ is the integral of CG; thus $\int dx = DH$, and $n\int dx = CG$. In the same manner as was found, $uu:zz::\int pdx:n\int pdx$, we have $u:z:: \sqrt{\int pdx} : \sqrt{n\int pdx}$. Putting into the proportion $t:T:: \int dx\sqrt{n\int pdx}: n\int dx\sqrt{\int pdx}$ the values DH, CG of $\int dx$ and $n\int dx$ and the ratio u:z in place of its equal $\sqrt{\int pdx}:\sqrt{n\int pdx}$, we have t:T::DH x z: CG x u. But by the construction we have DH:CG::BD:AC and we have found BD:AC::uu.zz: thus t:T::uuz: zzu::u:z. That is, the time used to reach the end of the space DH is to the time needed to reach the end of the space CG, as the velocity acquired at the end DH is to the velocity acquired at the end CG.[35]

It follows that the motion of the two balls is a uniformly accelerated motion, because the dead force or pressure of the equal balls L,P is equal; in the same way as their weight is equal, the spaces traversed are between them as the squares of their velocities. All this follows from the law of Galileo; but in this law when the spaces traversed are equal the times needed to traverse them are diminished and the impressions of gravity corresponding to these times are unequally diminished also. Since these impressions are equal only when the times are, the spaces are going to be augmented. Thus the force of elasticity which acts here in place of the pressures of gravity and by

[35] *Ibid.*, 43, 44. Deidier's symbols.

which the elastic springs are made to traverse the spaces equal to the bodies, make unequal impressions on the bodies. For example the first elastic unit, M, makes a greater impression on L than the second, and the second makes more than the third; the times corresponding to the slackening of equal springs are diminished. Thus while the twelve elastic units which act on ball L are equal, the impressions which they make on this ball are diminished in the degree that they are more distant. The same is true of the three elastic units acting on ball P. From this it follows that the impressions of the twelve units on ball L taken together are less than the forces of these twelve units taken together when the forces of the twelve elastic units are equal. Instead the impressions are diminished. For the same reason the impressions of the three elastic units which act on ball P taken together are less than the forces of these three units. But the "active forces" of the balls L and P are proportional to the elastic pressures which press them. Thus these living forces are less than the forces of the springs, and consequently they are not in the ratio of the spaces or the squares of the velocities. Bernoulli, he says, should have perceived the error in his own reasoning.[36]

To see that the forces of bodies in motion are here as

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[36] *Ibid.*, 45. 46.

the velocities of the same bodies, it is only necessary to consider that the velocities are as $\sqrt{12} : \sqrt{3}$ or $2\sqrt{3} : \sqrt{3}$ which is as 2 : 1. The time of the ball L is to the time of the ball P as 2 : 1.

That is why, supposing that two balls compress the elastics with the velocities acquired at the end of the expansions, the ball L will consume all its force only at the end of two units of time in each of which it will lose one unit of velocity. The ball P will lose its force at the end of the first unit of time. The velocity that it will lose was equal to the velocity that it had originally. But the ball L will continue its motion after the first time only because the velocity that it will have lost in compressing the elastics on its path is smaller in proportion to its total velocity. Likewise the velocity that the ball P will have lost in the same time is not great in proportion to its total velocity. By the contrary supposition the velocities removed from each ball in the same time are proportional to their total velocities, so the two balls would lose all their force in the first unit of time. It follows that the forces of these balls should be as the velocities lost in the same time, if the velocities lost in equal times were proportional to the velocities acquired. But the proportional portions of velocity that the balls would lose in the same time are as the velocities acquired, and not as their squares. Thus concludes Deidier, the forces of these balls are not as the squares of the acquired velocities, but as their simple velocities.[37]

The usual date cited for the conclusion of the controversy over the measure of force is 1743, the publication of d'Alembert's Treatise on Dynamics.[38] The controversy, however, lingered on for many years after this date. The several reasons accounting for this fact will be discussed in the "conclusion." However, one immediate reason is encountered on comparing the two editions of d'Alembert's "Treatise", the first edition of 1743 and the expanded and revised edition of 1758. The first edition accepts as valid measures of force, (A) the measure mdv for the case of equilibrium, /i.e. dead force/ which d'Alembert equates with quantity of motion, and (B) the measure mv^2 for the case of retarded motion where the "number of obstacles overcome" is as the square of the velocity. Here force is defined as "a term used to express an effect": (See copies of Preface 1743 ed.p~~xiix~~-xij, pp. 355-357.)

> Nevertheless as we have only the precise and distinct idea of the word force in restricting this term to express an effect I believe that the matter should be left to each to decide for himself as he wishes. The entire question cannot consist in more than a very futile metaphysical discussion or in a dispute of words unworthy of still occupying philosophers.[39]

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[37] Ibid., 47, 48.

[38] Jean d'Alembert, Traité de dynamique, 1st ed., Paris, 1743, preface. For references to the date 1743, see introduction to this dissertation p. 6.

[39] Ibid., xxj.

vement, ou l'on n'attache point d'idée nette au mot qu'on prononce, ou l'on ne peut entendre par-là en général, que la propriété qu'ont les Corps qui ſe meuvent, de vaincre les obſtacles qu'ils rencontrent, ou de leur réſiſter. Ce n'eſt donc ni par l'eſpace qu'un Corps parcourt uniformément, ni par le tems qu'il employe à le parcourir, ni enfin par la conſidération ſimple, unique & abſtraite de ſa maſſe & de ſa viteſſe qu'on doit eſtimer immédiatement la force, c'eſt uniquement par les obſtacles qu'un Corps rencontre, & par la réſiſtance que lui font ces obſtacles. Plus l'obſtacle qu'un Corps peut vaincre, ou auquel il peut réſiſter, eſt conſidérable, plus on peut dire que ſa *force* eſt grande, pourvû que ſans vouloir repréſenter par ce mot un prétendu être qui réſide dans le Corps, on ne s'en ſerve que comme d'une maniére abregée d'exprimer un fait, à peu près comme on dit, qu'un Corps a deux fois autant de *viteſſe* qu'un autre ; au lieu de dire qu'il parcourt en tems égal deux fois autant d'eſpace, ſans prétendre pour cela que ce mot de *viteſſe* repréſente un être inhérent au Corps.

Ceci bien entendu, il eſt clair qu'on peut oppoſer au Mouvement d'un Corps trois ſortes d'ob-

ſtacles ; ou des obſtacles invincibles qui anéantiſſent tout-à-fait ſon Mouvement, quel qu'il puiſſe être ; ou des obſtacles qui n'ayent préciſément que la réſiſtance néceſſaire pour anéantir le Mouvement du Corps, & qui l'anéantiſſent dans un inſtant, c'eſt le cas de l'équilibre ; ou enfin des obſtacles qui anéantiſſent le Mouvement peu à peu, c'eſt le cas du Mouvement retardé. Comme les obſtacles inſurmontables anéantiſſent également toutes ſortes de Mouvemens, ils ne peuvent ſervir à faire connoître la force : ce n'eſt donc que dans l'équilibre, ou dans le Mouvement retardé qu'on doit en chercher la meſure. Or tout le monde convient qu'il y a équilibre entre deux Corps, quand les produits de leurs maſſes par leurs viteſſes virtuelles, c'eſt-à-dire par les viteſſes avec leſquelles ils tendent à ſe mouvoir, ſont égaux de part & d'autre. (A) Donc dans l'équilibre le produit de la maſſe par la viteſſe, ou, ce qui eſt la même choſe, la quantité de Mouvement, peut repréſenter la force. Tout le monde convient auſſi que dans le Mouvement retardé, le nombre des obſtacles vaincus eſt comme le quarré de la viteſſe, enſorte qu'un Corps qui (B) a fermé un reſſort, par exemple, avec une certaine viteſſe, pourra avec une viteſſe double fermer, ou tout à la fois, ou ſucceſſivement, non pas deux, mais quatre reſſorts ſemblables au premier, neuf avec une viteſſe triple, & ainſi du reſte. D'où les partiſans des forces vives concluent que la force des Corps qui ſe meuvent actuellement, eſt en général comme le produit de la maſſe par le quarré de la viteſſe. Au fond, quel inconvénient pourroit-il y avoir, à ce que la meſure des forces fût différente dans l'équilibre & dans le Mouvement retardé, puiſque, ſi on veut ne raiſonner que d'après des idées claires, on doit n'entendre par le mot de *force*, que l'effet produit en ſurmontant l'obſtacle ou en lui réſiſtant? Il faut avouer cependant, que l'opinion de ceux qui regardent la force comme le produit de la maſſe par la viteſſe, peut avoir lieu nonſeulement dans le cas de l'équilibre, mais auſſi dans celui du Mouvement retardé, ſi dans ce dernier cas on meſure la force, non par la quantité abſolue des obſtacles, mais par la ſomme des réſiſtances de ces mêmes obſtacles. Car on ne ſauroit douter que cette ſomme de réſiſtances, ne ſoit proportionnelle à la quantité de Mouvement, puiſque, de l'aveu de tout le monde, la quantité de Mou-

vement que le Corps perd à chaque inſtant, eſt proportionnelle au produit de la réſiſtance par la durée infiniment petite de l'inſtant, & que la ſomme de ces produits, eſt évidemment la réſiſtance totale. Toute la difficulté ſe réduit donc à ſavoir ſi on doit meſurer la force par la quantité abſolue des obſtacles, ou par la ſomme de leurs réſiſtances. Il me paroîtroit plus naturel de meſurer la force de cette derniére maniére; car un obſtacle n'eſt tel qu'en tant qu'il réſiſte, & c'eſt, à proprement parler, la ſomme des réſiſtances qui eſt l'obſtacle vaincu: d'ailleurs, en eſtimant ainſi la force, on a l'avantage d'avoir pour l'équilibre & pour le Mouvement retardé une meſure commune: néanmoins comme nous n'avons d'idée préciſe & diſtincte du mot de *force*, qu'en reſtraignant ce terme à exprimer un effet, je crois qu'on doit laiſſer chacun le maître de ſe décider comme il voudra là-deſſus; & toute la queſtion ne peut plus conſiſter, que dans une diſcuſſion Métaphyſique très-futile, ou dans une diſpute de mots plus indigne encore d'occuper des Philoſophes. Auſſi n'auroit-elle pas ſans doute enfanté tant de volumes, ſi on ſe fut attaché à diſtinguer ce qu'elle renfermoit de clair & d'obſcur. En s'y d'Alembert, Traité, 1743

prenant ainsi, on n'auroit eu besoin que de quelques lignes pour décider la question : seroit-ce là ce que la plûpart de ceux qui ont traité cette matiére, auroient voulu éviter?

Discussions on living force and on statics and mechanics inserted here.

Après avoir donné au Lecteur une idée générale de l'objet que je me suis proposé dans cet Ouvrage, il ne me reste plus qu'un mot à dire sur la forme que j'ai cru devoir lui donner. J'ai tâché dans ma premiére Partie de mettre, le plus qu'il m'a été possible, les Principes de la Méchanique à la portée des commençans; je n'ai pu me dispenser d'employer le calcul différentiel dans la Theorie des Mouvemens variés; c'est la nature du sujet qui m'y a contraint. Au reste, j'ai fait ensorte de renfermer dans cette premiére Partie un assez grand nombre de choses dans un fort petit espace, & si je ne suis point entré dans tout le détail que la matiére pourroit comporter, c'est qu'uniquement attentif à l'exposition & au développement des Principes essentiels de la Méchanique, & ayant pour but de réduire cet Ouvrage à ce qu'il peut contenir de nouveau en ce genre, je n'ai pas crû devoir le grossir d'une infinité de propositions particuliéres que l'on trouvera aisément ailleurs.

La seconde Partie, dans laquelle je me su[...]
posé de traiter des loix du Mouvement des [...]
entr'eux, fait la portion la plus considéra[...]
l'Ouvrage : c'est la raison qui m'a engagé à [...]
ner à ce Livre le nom de *Traité de Dyn*[...]
Ce nom, qui signifie proprement la Scien[...]
puissances ou causes motrices, pourroit [...]
d'abord ne pas convenir à cet Ouvrage, d[...]
quel j'envisage plutôt la Méchanique com[...]
Science des effets, que comme celle des c[...]
néanmoins comme le mot de *Dynamique* [...]
usité aujourd'hui parmi les Savans, pour [...]
la Science du Mouvement des Corps, qui [...]
les uns sur les autres d'une maniére quelc[...]
j'ai cru devoir le conserver, pour annon[...]
Geométres par le titre même de ce Traité, [...]
m'y propose principalement pour but de [...]
tionner & d'augmenter cette partie de la [...]
nique. Comme elle n'est pas moins curieu[...]
le est difficile, & que les Problêmes qui s[...]
portent composent une classe très-étend[...]
plus grands Geométres s'y sont appliqués [...]
liérement depuis quelques années : mais i[...]
résolus jusqu'à présent qu'un très-petit no[...]
Problêmes de ce genre, & seulement [...]

To the second edition is added a section in which a third meaning is given to the measure of force. Here the valid measures of force are described as being (1) dead force, (2) the space traversed up to the total extinction of motion (<u>mv</u>^{2}) and (2) the space traversed uniformly in a given time (<u>mv</u>). (See copies of Preface, 1758 ed., pp.361-363.)

Thus the accepted solution to the measure of force problem was not clearly enunciated until 1758, although the usual date cited by historians is 1743. After an examination of the content of d'Alembert's preface to the "Treatise", other reasons will be suggested as to why the controversy did not cease with its publication.

In his preface to the *Traité de dynamique*, d'Alembert stated that he would only consider the motion of a body as the traversal of a certain space for which it uses a certain time. He rejected the idea of discussing the causes of motion and the inherent forces of moving bodies as obscure and metaphysical. It was for this reason, he said, that he refused to enter into an examination of the question of living forces. The question of causes is useless to mechanics. Mentioning in passing the part played by Leibniz, Bernoulli, Maclaurin and a lady famous for her spirit /Madame du Châtelet7, d'Alembert proposed to expose succintly the principles necessary to resolve the question.[40]

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[40] *Ibid*., xvi, xvii.

It is not by the space uniformly traversed by a body, nor the time needed to traverse it nor by the simple consideration of the abstract mass and velocity by which force should be estimated. Force should be estimated solely by the obstacles which a body encounters and by the resistance it makes to these obstacles. The greater the obstacles it can overcome or resist the greater is its force, provided that by the word force one does not mean something residing in the body.

One can oppose to the motion of a body three kinds of obstacles. First, obstacles which can completely annihilate its motion, second, obstacles which have exactly the resistance necessary to halt its motion, annihilating it for an instant as in the case of equilibrium, and third, obstacles which annihilate its motion little by little as in the case of retarded motion. Since the insurmountable obstacles annihilate all motion they cannot serve to make the force known. One must look for the measure of the force either in the case of (A) equilibrium or (B) in that of retarded motion.[41]

> Concerning these two possibilities for a measure, everyone agrees that there is equilibrium between two bodies, when the products of their masses by their virtual velocities, that is, the velocities by which they tend to move, are equal. Thus in the case of equilibrium the product of the mass by its velocity, or what is the same thing, the quantity of motion, can represent the force. Everyone agrees also that in retarded motion the number of obstacles overcome is as the square of the velocity. For example,

[41] *Ibid.*, xix.

> a body which closes one spring with a certain velocity can with a double velocity close, all together or successively, not two but four springs similar to the first, nine with a triple velocity etc.[42]

Thus here the force of a body is as the product of the mass by the square of the velocity. Should not then the word force mean only the effect produced in surmounting an obstacle or resisting it? Force should be "measured by the absolute quantity of the obstacles or by the sum of their resistances." Thus we have the precise and distinct idea of the word force as a term to express an effect.

The discussion concludes with the much quoted statement that "the question cannot consist in more than a completely futile metaphysical question, or a dispute over words unworthy of still occupying philosophers."[43]

At this point in the 1758 edition of the Traité de dynamique[44] is inserted what the forward to that edition described as "several reflections on the question of living forces","added to the preliminary discourse." In this insert three, rather than two, meanings of force are described. (see copies of preface to 1758 edition, pp. 361-363.)

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[42] Ibid., xix-xx.

[43] Ibid., xxj.

[44] Jean d'Alembert, Traité de dynamique, 1758, ed (Paris, 1921), XXX, This second edition was expanded and revised by d'Alembert. The explanation below was added to this edition. See copies of preface, 1758 ed.

rale des vents (1747), les *Recherches sur la précession des équinoxes et sur la nutation de l'axe de la terre* (1749), les *Eléments de musique théorique et pratique, suivant les principes de M. Rameau, éclaircis, développés et simplifiés* (1752), l'*Essai d'une nouvelle théorie sur la résistance des fluides* (1752), les *Recherches sur les différents points importants du système du monde* (1754-56), les *Eléments de philosophie* (1759), et enfin les *Opuscules mathématiques* (8 vol. in-4°, 1761-1780), qui contiennent des mémoires où sont traités des sujets du domaine de l'analyse, de la mécanique et de l'astronomie. A ces travaux il faut encore ajouter les *Nouvelles tables de la lune*, la *Nova tabularum lunarium emendatio*, le *Discours préliminaire* (1751) et les nombreux articles de l'*Encyclopédie*, les *Mélanges de littérature, d'histoire et de philosophie* (1759-60), et ses *Eloges académiques*.

D'Alembert fit preuve dans ses recherches d'une vigueur d'esprit peu commune, et fut souvent obligé d'inventer de toutes pièces de nouvelles méthodes analytiques pour donner une solution satisfaisante aux problèmes les plus ardus : c'est ainsi qu'il a créé la théorie des équations aux dérivées partielles pour résoudre le problème des cordes vibrantes. Par ses travaux de mécanique il a préparé la voie à la *Mécanique analytique* de Lagrange, et entre Newton et Laplace il occupe la position la plus élevée dans la mécanique céleste.

Penseur autant que géomètre, original et profond, il n'a jamais perdu de vue dans ses investigations l'intérêt philosophique que présentent les problèmes, et ses écrits abondent en vues pénétrantes sur les fondements des sciences, les principes de la connaissance et les bases de la morale. Parmi les mathématiciens du XVIII^{e} siècle, il est celui qui est le plus pénétré de philosophie.

A ces éminentes qualités de savant il joignait celles d'une grande bonté de cœur et d'un très vif sentiment de la justice. Les exemples de bienfaisance et de dévouement qu'il a donnés pendant sa vie trouvent leur expression adéquate dans cette belle pensée qu'il a écrite dans ses *Eléments de philosophie* : le désintéressement est la première des vertus morales, et il n'est pas permis d'avoir du superflu, lorsque d'autres hommes n'ont pas même le nécessaire.

M. S.

1758 edition of Traité de dynamique
Jean d'Alembert

AVERTISSEMENT

Cette seconde édition est augmentée de plus d'un tiers.

On a ajouté au discours préliminaire quelques réflexions sur la question des forces vives, et l'examen d'une autre question importante, proposée par l'Académie Royale des Sciences de Prusse, Si les lois de la Statique et de la Mécanique sont de vérité nécessaire ou contingente ?

Dans la première Partie de l'Ouvrage, ce qui regarde la mesure et la comparaison des forces accélératrices est expliqué avec beaucoup plus de détail que dans la première Edition, et contient sur cette matière des remarques qu'on ne trouvera point ailleurs; on a inséré aussi, dans cette première Partie, plusieurs nouvelles recherches sur les lois de l'équilibre.

Les additions principales de la seconde Partie sont quelques propositions sur l'état du centre de gravité de plusieurs Corps qui agissent les uns sur les autres ; la solution complète d'un Problème de Dynamique, qui n'avait été qu'imparfaitement résolu jusqu'ici, parce qu'on n'avait pu séparer les indéterminées de l'équation finale anéantissent tout à fait son Mouvement, quel qu'il puisse être; ou des obstacles qui n'aient précisément que la résistance nécessaire pour anéantir le Mouvement du Corps, et qui l'anéantissent dans un instant, c'est le cas de l'équilibre; ou enfin des obstacles qui anéantissent le Mouvement peu à peu, c'est le cas du Mouvement retardé. Comme les obstacles insurmontables anéantissent également toutes sortes de Mouvements, ils ne peuvent servir à faire connaître la force : ce n'est donc que dans l'équilibre, ou dans le Mouvement retardé qu'on doit en chercher la mesure. Or tout le monde convient qu'il y a équilibre entre deux corps, quand les produits de leurs masses par leurs vitesses virtuelles, c'est-à-dire par les vitesses avec lesquelles ils tendent à se mouvoir, sont égaux de part et d'autre. Donc dans l'équilibre le produit de la masse par la vitesse, ou, ce qui est la même chose, la quantité de Mouvement, peut représenter la force. Tout le monde convient aussi que dans le Mouvement retardé, le nombre des obstacles vaincus est comme le carré de la vitesse; en sorte qu'un Corps qui a fermé un ressort, par exemple, avec une certaine vitesse, pourra avec une vitesse double fermer, ou tout à la fois, ou successivement, non pas deux, mais quatre ressorts semblables au premier, neuf avec une vitesse triple, et ainsi du reste. D'où les partisans des forces vives concluent que la force des Corps qui se meuvent actuellement, est en général comme le produit de la masse par le carré de la vitesse.

(A) = 1743 edition

(B) = 1743 edition

Au fond, quel inconvénient pourrait-il y avoir à ce que la mesure des forces fût différente dans l'équilibre et dans le Mouvement retardé, puisque, si on veut ne raisonner que d'après des idées claires, on doit n'entendre par le mot de *force*, que l'effet produit en surmontant l'obstacle ou en lui résistant ? Il faut avouer cependant que l'opinion de ceux qui regardent la force comme le produit de la masse par la vitesse, peut avoir lieu non seulement dans le cas de l'équilibre, mais aussi dans celui du Mouvement retardé, si dans ce dernier cas on mesure la force, non par la quantité absolue des obstacles, mais par la somme des résistances de ces mêmes obstacles. Car on ne saurait douter que cette somme de résistances ne soit proportionnelle à la quantité de Mouvement, puisque, de l'aveu de tout le monde, la quantité de Mouvement que le Corps perd à chaque instant, est proportionnelle au produit de la résistance par la durée infiniment petite de l'instant, et que la somme de ces produits est évidemment la résistance totale. Toute la difficulté se réduit donc à savoir si on doit mesurer la force par la quantité absolue des obstacles, ou par la somme de leurs résistances. Il paraîtrait plus naturel de mesurer la force de cette dernière manière, car un obstacle n'est tel qu'en tant qu'il résiste, et c'est, à proprement parler, la somme des résistances qui est l'obstacle vaincu : d'ailleurs, en estimant ainsi la force, on a l'avantage d'avoir pour l'équilibre et pour le Mouvement retardé une mesure commune; néanmoins comme nous n'avons d'idée précise et distincte du mot de *force*, qu'en restreignant ce terme à exprimer un effet, je crois qu'on doit laisser chacun le maître de se décider comme il voudra là-dessus, et toute la question ne peut plus consister, que dans une discussion Métaphysique très futile, ou dans une dispute de mots plus indigne encore d'occuper des Philosophes.

1743 edition

Addition

1758 edition

Tout ce que nous venons de dire suffit assez pour le faire sentir à nos Lecteurs. Mais une réflexion bien naturelle achèvera de les en convaincre. Soit qu'un Corps ait une simple tendance à se mouvoir avec une certaine vitesse, tendance arrêtée par quelque obstacle; soit qu'il se meuve réellement et uniformément avec cette vitesse; soit enfin qu'il commence à se mouvoir avec cette même vitesse, laquelle se consume et s'anéantisse peu à peu par quelque cause que ce puisse être : dans tous ces cas, l'effet produit par le Corps est différent, mais le corps considéré en lui-même n'a rien de plus dans un cas que dans un autre; seulement l'action de la cause qui produit l'effet est différemment appliquée. Dans le premier cas, l'effet se réduit à une simple tendance, qui n'a point proprement de mesure précise, puisqu'il n'en résulte aucun mouvement; dans le second, l'effet est l'espace parcouru uniformément dans un temps donné, et cet effet est proportionnel à la vitesse; dans le troisième, l'effet est l'espace parcouru jusqu'à l'extinction totale du Mouvement, et cet effet est comme le carré de la vitesse. Or ces différents effets sont

évidemment produits par une même cause; donc ceux qui ont dit que la force était tantôt comme la vitesse, tantôt comme son carré, n'ont pu entendre parler que de l'effet, quand ils se sont exprimés de la sorte. Cette diversité d'effets provenant tous d'une même cause, peut servir, pour le dire en passant, à faire voir le peu de justesse et de précision de l'axiome prétendu, si souvent mis en usage, sur la proportionnalité des causes à leurs effets.

Enfin ceux mêmes qui ne seraient pas en état de remonter jusqu'aux Principes métaphysiques de la question des forces vives, verront aisément qu'elle n'est qu'une dispute de mots, s'ils considèrent que les deux partis sont d'ailleurs entièrement d'accord sur les principes fondamentaux de l'équilibre et du mouvement. Qu'on propose le même Problème de Mécanique à résoudre à deux Géomètres, dont l'un soit adversaire et l'autre partisan des forces vives, leurs solutions, si elles sont bonnes, seront toujours parfaitement d'accord; la question de la mesure des forces est donc entièrement inutile à la Mécanique, et même sans aucun objet réel. Aussi n'aurait-elle pas sans doute enfanté tant de volumes, si on se fût attaché à distinguer ce qu'elle renfermait de clair et d'obscur. En s'y prenant ainsi, on n'aurait eu besoin que de quelques lignes pour décider la question, mais il semble que la plupart de ceux qui ont traité cette matière, aient craint de la traiter en peu de mots.

La réduction que nous avons faite de toutes

The three cases are: (1) /dead force/ where a body has a tendency to move itself with a certain velocity, but the tendency is arrested by some obstacle. (2) /quantity of motion/ in which the body actually moves uniformly with this certain velocity. (3) /living force/ where the body moves with a velocity which is consumed and annihilated little by little by some cause. The effect produced in each case is different, because in each the action of the same cause is differently applied. The body in itself however possesses nothing more in one case than the other. "In the first case the effect is reduced to a simple tendency which is not properly a measure since no motion is produced. In the second *the effect is the space traversed uniformly in the given time* and this effect is proportional to the velocity. In the third case, *the effect is the space traversed up to the total extinction of motion*, and this effect is as the square of the velocity.[45]

The two parties, concluded d'Alembert, are entirely in accord over the fundamental principles of equilibrium and motion, and their solutions are in perfect agreement. Thus the question is a "dispute over words" and is "entirely futile for mechanics."

Thus although the 1743 edition of d'Alembert's *Treatise* had been cited by many authors as resolving the dispute, it

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[45]*Ibid.*, xxx, italics added.

provided little more clarification than contrasting dead with living forces and calling the argument a "dispute over words." 'S Gravesande in 1729 had also called it a dispute over words but neither he nor d'Alembert in 1743 really defined in what way this was true. The date 1743 then has little significance as a terminus for the controversy.

Although the 1758 edition of the Treatise actually did point out the validity of both measures of force, d'Alembert was likewise anticipated in this insight by Roger Boscovich. It was Boscovich's De Viribus Vivis published in Rome in 1745 which furnished the essential insight establishing the separate sphere of application for both measures of force.[46] This is a work of fifty pages of difficult latin which deals with two separate subjects involving living force. It shows that Boscovich possessed a very complete background of the history of the quarrel before his own intervention, from Leibniz and Bernoulli to Voltaire, de Mairan, and du Châtelet. He does not cite d'Alembert's Treatise on Dynamics but this had been published only two years earlier.[47]

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[46] The ensuing discussion very closely follows an excellent account of the contents of the De Viribus Vivis written by Pierre Costabel: "Le De Viribus Vivis de R. Boscovic ou de la vertu des querelles de mots," Archives Internationales d'Histoire des sciences (1961) 3-12.

[47] Here Costabel recognizes that d'Alembert's own part in the controversy did not really occur until the 1758 edition of the Treatise. However he does not specify that here

He did not meet d'Alembert until a visit to Paris in 1759. Nor does he mention Euler whose Mechanica of 1736 contained ideas suggestive of a general treatment of mechanics. The reflections of Euler on the nature of forces did not take form until 1749-1750.

Employing both the ancient scholastic categories and the new mathematical methods of his time, Boscovich was able to show the nature of force as it was applied over a distance and through a time by means of graphs. Vis Activa for Boscovich which was identical to Leibniz's vis mortua, was the "instantaneous action" by which the power (puissance) passes into action and engenders a new velocity. This instantaneous pressure passes to a velocity not by multiplication of effects in the course of an instant but only by continuous application. In the same way a line gives a surface not by its own multiplication but by its continual path following another line. A pressure is connected as a straight line to the surface engendered. The power (puissance) passes into action not by multiplication of effects but by generating a being of two dimensions of which geometry is the only means of rendering it adequately.

Thus without taking a position on the definition of

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d'Alembert added the section to the preface concerning the difference between a force acting through a time and a force acting over a distance. He indicates rather that this was due to d'Alembert's addition of a section generalizing the principle of living force added to the main body of the Treatise. See Costabel, p. 4.

force, Boscovich is measuring a characteristic of the velocity acquired, by a ratio composed of the pressure and its duration; by a geometric image. This image is the surface generated by a line representing the pressure when time is the second dimension of a diagram. (When the pressure is contant, the case is that of gravity.)[48]

Now if the time coordinate is replaced by the space traversed and the pressure coordinate by that of the "forces which in each instant generate the velocities which are proportional to them", a second aspect of the concept of "force" is represented.[49]

On the question of elastic and inelastic collision Boscovich systematized the principle of action and reaction and its equivalent: the conservation of quantity of motion taken in an algebraic sense. In verifying the conservation of living force in the sense of Leibniz, he said that living force being formed as it is by the square of the velocity destroys the sign of that velocity, whereas the quantity of motion conserves all its characteristic elements.

Boscovich concludes in paragraph 39 of the De Viribus Vivis that the question of living force is a question of language and completely useless.[50]

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[48] Ibid., 6, 7.

[49] Ibid., 7. For the translation of Boscovich's geometric discussion into algebraic terminology see Costabel, pp. 8,9.

[50] Ibid. 9.

In spite of this analysis of "force" however Boscovich believed that momentum was the true measure of force, vis viva being valid only as a method of calculation. He discussed this problem in his Philosophiae naturalis theoria (1758). His analysis is discussed in detail by Thomas Hankins who writes:

> [Boscovich] believed he had caught the defenders of Leibniz in an error and wrote a rather confused section of his Theoria where he tried to prove that no 'force of motion' is contained in a moving body by impact. The only forces are those mutually acting 'dead forces' that arise when bodies collide. Nothing is passed from one body to the next and no active force or vis viva exists in a moving object. He could not deny that the quantity mv^2 is conserved in elastic collisions, but he did deny that this quantity represented any real thing.[51]

Indeed in his Theory of Natural Philosophy, Boscovich wrote:

> ...it will be sufficiently evident, both from what has already been proved as well as from what is to follow, that there is nowhere any sign of such living forces nor is this necessary. For all the phenomena of Nature depend upon motions and equilibrium, and thus from dead forces and the velocities induced by the action of such forces. For this reason, in the dissertation De Viribus Vivis, which was what led me to this theory thirteen years ago, I asserted that there are no living forces in Nature...[52]

Thus Boscovich while providing an insight which theoretically helped to resolve the vis viva controversy

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[51] Thomas Hankins, "Eighteenth-Century Attempts to Resolve the Vis Viva Controversy," Isis 56 (1965) 292. Boscovich's ideas on living force are discussed by Hankins on pp. 291-297.

[52] Roger Boscovich, A Theory of Natural Philosophy, trans. J. M. Child from the second edition (1763), London, 1922, 293. Quoted in Hankins, p. 292.

did not claim equal status for the two principles in treating physical problems.

Subsequent contributions to the controversy indicated that confusion and discussions over the measure of force existed through the remainder of the decade. Indeed it is not possible to decide when both measures of forces were accepted as valid (see appendix).

An article by James Jurin in 1745 proposed an *experimentum crucis* to decide the issue dividing the Leibnizian and Cartesian camps.[53] This experiment was similar to the one he had earlier proposed to the Royal Society and which had been described and analyzed by Madame du Châtelet (see this dissertation, pp.332-333. Jurin does not disucss Du Châtelet's comments). Jurin drew his analysis from the action of compressed springs. He categorized two types of mechanical forces: (1) The force of a body at rest, called pressure, tension, or *vis mortua*, as exemplified by a body lying on a table, hanging by a rope, or supported on a spring. (2) The force of a body in motion, this being a power in the body by which it can remove obstacles in its way or lessen, destroy, or overcome the force of another body and which bodies in turn can alter its own force. It is whether the measure of this moving force is *mv* or mv^2 that is in dispute.

[53] James Jurin, "An Inquiry into the Measure of the Force of Bodies in Motion? With a Proposal of an *Experimentum Crucis* to Decide the Controversy About it," (1745) *Phil. Trans.*, 43, 423-440.

In regard to the action of a compressed spring accelerating a body he writes:

> ...all the force which resided in the spring while bent, is now upon the unbending of the spring, communicated to the body moved. I ask therefore, what was that force, or what kind of force was that which resided in the spring, while bent and without motion? Was it a bare pressure or a moving force? You must acknowledge it was a vis mortua, a bare pressure, and nothing more. But the force communicated to the body, and which now resides in the body in motion, is a vis viva, a moving force. This therefore is not the same force nor a force of the same kind as resided in the bent spring.[54]

His crucial experiment to decide the issue showed that bodies which moved acquired an mv. From a particular case using compressed springs, Jurin drew the general conclusion that force is to be measured by mv. Thus in 1745 Jurin himself said:

> ...tho' both Parties agree in the event of the experiments; yet as the writers on each side have found a way of deducing from those experiments a conclusion suitable to their own opinion the disagreement still continues as wide as ever to the great scandal of the learned world.[55]

Immanual Kant's first published work "Gedanken von der wahren Schätzung der lebendigen Kräfte" of 1746 reviews the arguments of such men as Leibniz, Papin, Bernoulli, Jurin, Herman, Bulfinger, de Mairan and du Châtelet.[56] In addition

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[54] Ibid., 431.

[55] Ibid., 426.

[56] Immanual Kant, "Gedanken von der wahren Schätzung der lebendigen Kräfte," first published Königsberg, 1746, in Immanual Kant's Werke, Berlin, 1922, 1, 1-187.

it brings out certain philosophical issues of interest mainly to the evolution of Kant's thought.

Concerning this work Max Jammer writes:

> In his Thoughts on the True Estimation of Living Forces, Kant steers a middle course between the Cartesians and the Leibnizians in their dispute about the true measure of force, and in his Metaphysical Foundations of Natural Science, written after his precritical period, Kant aims at a philosophical foundation of Newtonian physics, while some of his preconceptions still exhibit a strong influence of Leibniz and Wolff.
>
> The major objective in Kant's Thoughts on the True Estimation of Living Forces is of little concern for us at present. By an erroneous classification of motions into two kinds, one that persists in the body to which it was communicated and continues indefinitely, and one that ceases with the cessation of the external force that produces it, Kant attempts to do justice to both the Cartesians and Leibnizians; the Leibnizian measure of mv^2, according to Kant, applies to forces producing motions of the first type, whereas the Cartesian measure applies to forces producing motions of the second type. Kant accepts the Leibnizian concept of living force as essential to matter and agrees with Leibniz's dictum: "Est aliquid praeter extensionem, imo extensione prius." And like Boscovich he comes to the conclusion that "it is easily proved that there would be no space and no extension if substances had not force whereby they can act outside themselves. For without a force of this kind there is no connection, without this connection no order and without this order no space." It is important to note that in this work, force is for Kant the most fundamental concept and basic for further inferences. However in his critical period the order of these dependences is reversed and the concept force appears at the end of the chain of inferences.[57]

An article by Daniel Bernoulli entitled "Remarks on the Principle of the Conservation of Living Forces Taken in

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[57] Max Jammer, Concepts of Force, Cambridge, Mass, 1957, 179.

a General Sense," (1748)[58] derived a general expression for the conservation of mv^2 when due to diversity in the positions of bodies, uniform gravitation is altered either in respect to intensity or direction. Here the living force can no longer be estimated by the descent of the center of gravity multiplied by the mass.

Conclusion

The controversy over living force in the 1740's began with the particular issues dividing Mairan and Madame du Châtelet and moved on to more general clarifications on the nature and causes of the controversy itself.

Du Châtelet showed that the effects produced by uniform and by retarded motion are different and cannot be compared. For real obstacles actually overcome in retarded motion it is not possible to substitute the imaginary situation of uniform motion in which these obstacles are not surmounted. The actual physical situation at hand must be the one discussed.

Mairan recognized the error in the sign of the velocity in mv, which due to Descartes' faulty formulation of $m|v|$ conservation had been a stumbling block in impact problems.

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[58] Daniel Bernoulli, "Remarques sur le principe de la conservation des forces vives pris dans un sens general," Histoire de l'academie royale des sciences et belles lettres de Berlin, (1748) 356-364.

Voltaire's paper added confusion to the controversy by reiterating Mairan's unorthodox argument reducing retarded motion to uniform motion and by arguing for Descartes' conservation of $m|v|$.

Deidier like Mairan attempted to reduce the vis viva problem of free fall to a momentum problem by using the "force" a body would have if it were not moving under the restraining action of gravity. He also attempted to refute Bernoulli's spring argument by calculating and comparing the times during which the two springs expand over the given spaces. This calculation showed the times to be in the same ratio as the velocities of the bodies, establishing mv as a measure of their "force".

D'Alembert's 1743 edition of the Treatise discussed only dead and living forces, even though he called the controversy "a dispute over words". In his 1758 edition he included momentum in his analysis, establishing the validity and use of both mv and mv^2. In this insight however he was anticipated by Boscovich (1745). Thus the idea prevalent in the literature that d'Alembert "resolved" the controversy is not borne out by a careful examination of the evidence.

Other contributions by Jurin and Kant in the late 1740's indicated that confusion still existed over which quantity to call the measure of force. As will be shown by the appendix contributions to the controversy gradually died out, but both measures were not accepted on equal footing until early in the nineteenth century.