Canopy Modeling: Eulerian and Lagrangian Closure Schemes

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ESPM 228
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Spring, 2016
Outline

- Roles of Turbulence
- Eulerian Closure Schemes
- Lagrangian Turbulence Schemes
Wind and turbulence experience strong gradients in vegetation. How do we deal with this? We have to predict wind and turbulence profiles through the canopy.
Next we discuss turbulence in the canopy. We have two options. Lagrangian or Eulerian frameworks.

<table>
<thead>
<tr>
<th>Turbulence Closure Schemes</th>
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<tbody>
<tr>
<td><strong>Lagrangian</strong></td>
</tr>
<tr>
<td><strong>Eulerian</strong></td>
</tr>
<tr>
<td>– Zero Order, $c(z) =$constant</td>
</tr>
<tr>
<td>– First Order, $F = K \frac{dc}{dz}$</td>
</tr>
<tr>
<td>– Second Order and ++ $(dc/dt, dw’c’/dt)$</td>
</tr>
<tr>
<td>– Large Eddy Simulation, LES</td>
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</table>
Turbulence Frameworks

• Lagrangian, $dc/dt$
  – It is a moving framework that follows a trajectory of an ensemble of fluid parcels, like viewed from a drone that is traveling with a smoke plume

• Eulerian,
  – It is a static framework that studies the local derivative of $c$ as time passes a fixed point, like viewed from a tower. It has advective and higher order terms
Unique features of canopy turbulence that must be accounted for by a competent theory:

• momentum is absorbed by the ground and elevated plant elements

• a hydrodynamically unstable inflexion occurs in the wind profile at the canopy atmospheric interface.

• Turbulence in the vegetation is vertically inhomogeneous.

• mean kinetic energy is converted to turbulent kinetic energy in the wake of plants, which accelerates the eddy cascade.
Turbulence is quite complicated and is one of the great unsolved problems of physics. It is multiscaled yielding different answers at different scales.
Zero Order Closure Test

- Upper left: Average Daily E vs. Average Daily Rn
  - Slope: 0.271
  - R²: 0.881
  - R² (adjusted): 0.880

- Upper right: Average Daily LE vs. Average Daily Rn
  - Slope: 0.372
  - R²: 0.615
  - R² (adjusted): 0.606

- Lower right: Average Daily H vs. Average Daily Rs
  - Slope: 0.0119
  - R²: 0.064
  - R² (adjusted): 0.059

- Lower left: Average Daily G vs. Average Daily Rs
  - Slope: 0.25
  - R²: 0.42
  - R² (adjusted): 0.398
We face the central problem of closure because the budget equation for $c$, $dc/dt$, has one equation and two unknowns, $c$ and $w'c'$, the flux. First order closure solves the problem by applying K theory. This works fine above a canopy in the surface boundary layer. It has problems in canopy and within 2 times canopy height.
**K Theory in the Surface Layer**

\[ K_m(z) = \frac{w' u'(z)}{\rho \frac{\partial u}{\partial z}} \]

**Short vegetation**

\[ K_m(z) = \frac{u_r^2}{\frac{\partial u}{\partial z}} = \frac{u_r^2}{u_r / k z} = u_r k z \]

**Tall vegetation**

\[ K_m(z) = \frac{u_r^2}{\frac{\partial u}{\partial z}} = \frac{u_r^2}{u_r / k(z - d)} = u_r k(z - d) \]
K theory is ill advised in canopies because counter gradient transfer occurs, leading to negative K's which are non physical.
Many observations show strong shear in the canopy, secondary wind maxima, and kinks in scalar profiles, all evidence of non-local transport. Hence K theory tends to fail

Stanley Corrsin (1974) wrote that several conditions must hold to apply K-theory:

- the length scales of the turbulent transfer must be less than the length scales associated with the curvature of the concentration gradient of the scalar.
- the turbulent length scale must be constant over the distance where the concentration gradient changes significantly.

\[ w'c' \sim K_c \frac{\partial c}{\partial z} + TT \]
Counter Gradient ‘Diffusion’ from Non-Local Transport

\[
\frac{\partial \bar{c}}{\partial z} = \left[ \frac{w' c'}{\tau} - \frac{\partial (w' w' c')}{\partial z} \right] / w' w'
\]

\[
\overline{w' c'} = \tau \left( \overline{w'^2} \frac{\partial \bar{c}}{\partial z} + \frac{\partial \overline{w' w' c'}}{\partial z} \right)
\]
Theoreticians Who Advanced Higher Order Closure Modeling to Plant Canopies

Roger Shaw  John Finnigan  K T Paw II

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Second Order Budget

\[ u'u', w'w', v'v', w'u' \]

3rd Order Unknowns

\[ u'u'u', w'w'w', w'u'u', u'w'w' \]

closure

\[ w'w'u' = -K \frac{\partial w'u'}{\partial z} \]

2nd order closure tries a work around by defining budget equations for the second moment terms. But . If we solve for 2nd order terms, they produce equations with 3rd order unknowns

9 unknowns are associated with 2nd order budgets, \( u'w' \), \( u'u' \), \( v'v' \), \( w'w' \), \( w'u'w' \), \( w'u'u' \), \( w'v'v' \), \( w'w'w' \)
And third order moments produce 4th order terms
### Higher Order Closure Equations and Unknowns

<table>
<thead>
<tr>
<th>Order of Closure</th>
<th>Turbulence Budget Equations</th>
<th>Unknowns</th>
<th>Scalar Budget Equations</th>
<th>Unknowns</th>
<th>Closure Scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero</td>
<td></td>
<td>$u, w$</td>
<td>$\frac{DC}{Dt}$</td>
<td></td>
<td>$c = f(t, x, y, z)$</td>
</tr>
<tr>
<td>First Order</td>
<td>$u$</td>
<td>$w^2, \overline{w^3}, \overline{w^4}, \overline{w^5}$</td>
<td>$\overline{\tau} \cdot \overline{q} = \overline{C} \overline{T}$</td>
<td>$\overline{w}^2 = -K \frac{DC}{Dz}$</td>
<td></td>
</tr>
<tr>
<td>Second Order</td>
<td>$u'$ $w', \overline{w'^2}, \overline{w'^3}$</td>
<td>$\overline{w'}, \overline{w'^2}, \overline{w'^3}$</td>
<td>$\overline{p}' \cdot \overline{q}' = \overline{C} \overline{T}'$</td>
<td>Numerous third order moments</td>
<td>$\overline{w'}^2 = -K \frac{DC}{Dz}$</td>
</tr>
<tr>
<td>Third Order</td>
<td>$\overline{w'^2}, \overline{w'^3}, \overline{w'^4}$</td>
<td>$\overline{w'^2}, \overline{w'^3}, \overline{w'^4}$</td>
<td>$\overline{w'^2}, \overline{w'^3}, \overline{w'^4}$</td>
<td>Numerous fourth order moments</td>
<td>$\overline{w'^2}, \overline{w'^3}, \overline{w'^4}$</td>
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Instantaneous Navier Stokes Eq

Fluid Velocity Budget or Fluid Acceleration (force) equation

\[
\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = - \frac{\partial p}{\partial x_i} + \frac{g \theta}{T} \delta_{i3} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}
\]
Einstein Notation

Summation implied with repeated indices

\[ a_{ij} A_j = a_{i1} A_1 + a_{i2} A_2 + a_{i3} A_3 \]

\[ u_j \frac{\partial u_i}{\partial x_j} = u_1 \frac{\partial u_i}{\partial x_1} + u_2 \frac{\partial u_i}{\partial x_2} + u_3 \frac{\partial u_i}{\partial x_3} \]
Expansion of Total Derivative

\[
\frac{dc(t,x,y,z)}{dt} = \frac{\partial c}{\partial t} + \frac{dx}{dt} \frac{\partial c}{\partial x} + \frac{dy}{dt} \frac{\partial c}{\partial y} + \frac{dz}{dt} \frac{\partial c}{\partial z} + u \frac{\partial c}{\partial x} + v \frac{\partial c}{\partial y} + w \frac{\partial c}{\partial z}
\]
Mean Budget Equations and Reynolds Averaging

Time Partial Derivative

\[
\frac{\partial \bar{u}_i}{\partial t} = \frac{\partial (\bar{u} + u')}{\partial t} = \frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t} = \frac{\partial \bar{u}}{\partial t}
\]
Advection Term

\[
\frac{u_j}{x_j} \frac{\partial u_i}{\partial x_j} = (u + u') \left( \frac{\partial u}{\partial x} + \frac{\partial u'}{\partial x} \right) = \\
- \frac{\partial u}{\partial x} + u' \frac{\partial u}{\partial x} - u \frac{\partial u'}{\partial x} + u \frac{\partial u'}{\partial x} \\
- \frac{\partial u}{\partial x} + u' \frac{\partial u'}{\partial x}
\]
Time Averaging and 2\textsuperscript{nd} order Terms

\[ u_j \frac{\partial u_i}{\partial x_j} \]

\[ \frac{\partial \bar{u}_i' u_j'}{\partial x_j} = u_i' \frac{\partial \bar{u}_j'}{\partial x_j} + u_j' \frac{\partial \bar{u}_i'}{\partial x_j} \]

\text{Continuity Eq}

\[ \frac{\partial u_j'}{\partial x_j} = 0 = \frac{\partial u_j }{\partial x_j} = \frac{\partial \bar{u}_j }{\partial x_j} \]

\[ \frac{\partial \bar{u}_i' u_j'}{\partial x_j} = u_j' \frac{\partial \bar{u}_i'}{\partial x_j} \]

\text{New term arises at 2\textsuperscript{nd} order:}

\text{Crux of the turbulence closure problem}
Pressure Gradient

\[
\frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial x_i} + \frac{\partial p'}{\partial x_i}
\]
Viscous Drag

\[ \nu \frac{\partial^2 u}{\partial x_j \partial x_j} = \nu \frac{\partial^2 u}{\partial x_j \partial x_j} + \nu \frac{\partial^2 u'}{\partial x_j \partial x_j} \]

\[ \nu \frac{\partial^2 u'}{\partial x_j \partial x_j} \approx 0 \]
Simplified Mean Budget for $u$

$$\frac{\partial \bar{u}}{\partial t} = 0 = \frac{\partial \bar{u}^l}{\partial z} - \frac{\partial \bar{p}^l}{\partial x} - \frac{\partial \bar{p}}{\partial x} + \nu \frac{\partial^2 \bar{u}}{\partial x^2}$$
2nd Order Equation

\[
\frac{\partial u_i^j}{\partial t} + u_j \frac{\partial u_i^j}{\partial x_j} = u_i \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i^j u_i^j}{\partial x_j} - \\
\frac{\partial u_i^j p^j}{\partial x_i} = u_i \frac{\partial p}{\partial x_i} - u_j \frac{\partial p}{\partial x_i} + \rho \left( \frac{\partial u_i^j}{\partial x_i} + \frac{\partial u_i^j}{\partial x_j} \right) + \rho \left( \frac{\partial u_i^j}{\partial x_i} + \frac{\partial u_i^j}{\partial x_j} \right) + u_i \frac{\partial u_i^j}{\partial x_i} \frac{\partial u_i^j}{\partial x_j} - 2u_i \frac{\partial u_i^j}{\partial x_j} \frac{\partial u_i^j}{\partial x_j}
\]
TKE Budget

\[ \bar{e} = \frac{1}{2} (u'u' + v'v' + w'w') \]

dissipation

\[ \frac{\partial \bar{e}}{\partial t} = 0 = -w' u' \frac{\partial \bar{u}}{\partial z} + \frac{g w' \theta}{\theta} \frac{\partial \bar{w'}}{\partial z} - \frac{\partial \bar{w'} e'}{\partial z} - \frac{\partial p' w'}{\partial \rho \partial z} - \varepsilon \]

Shear Production

Pressure Production

Return to Isotropy Via P-V Interactions
Variance Budget

\[
\frac{\partial \overline{u' u'}}{\partial t} = 0 = -2w'u' \frac{\partial \overline{u}}{\partial \eta} - \frac{\partial w'u' u'}{\partial \eta} - 2 \frac{u' \partial p'}{\partial \overline{\rho} \partial \chi} - 2 \frac{u' \partial p'}{\partial \overline{\rho} \partial \chi} - \frac{2\varepsilon}{3}
\]

Shear Production

Pressure Interaction
Specific 2nd order budget equations for turbulence variance and covariances, including Reynolds stress and water vapor. There are shear production terms, non local transport and pressure interaction terms.
Higher order closure models are one way to work around the problems with K theory. These can be derived starting with Navier Stokes and applying Reynolds averaging rules. Nothing magic. But assumptions needed to solve for terms like dp'/dx
Pressure fluctuation is a function of the drag coefficient

\[ \frac{\partial p^1}{\partial x} = C_d a(z) \bar{u}^2 \]

\[ \frac{\partial \bar{u}' \bar{w}'}{\partial z} = C_d a(z) \bar{u}^2 \]

\[ \bar{u}(z) = u_h \exp\left(\alpha\left(1 - \frac{z}{h}\right)\right) \]
Flux Conservation Budget:
What the Terms Mean

\[
\frac{\partial w' q'}{\partial t} = 0 = -w' w' \frac{\partial q}{\partial z} - \frac{\partial w' w' q'}{\partial z} - \frac{q' \partial p'}{\partial z} + \beta \frac{\partial T}{\partial y} q'
\]

Non-local Transport
Buoyancy prod

Shear Prod
Pressure Interactions
Form of Third Order Budget Equation

\[ \frac{\partial a' a' b'}{\partial t} = 0 = -w' a' a' \frac{\partial h}{\partial z} - 2 w' a' b' \frac{\partial a}{\partial z} - w' b' \frac{\partial a' a'}{\partial z} - 2 w' a' \frac{\partial a' b'}{\partial z} - \epsilon_{mb} \]
Simulation of Turbulence Statistics

\( \langle u'w' \rangle \)

\( \langle u'u' \rangle \)

\( \langle u \rangle \)

\( \langle w'u' \rangle \)

\( \langle u'w'w' \rangle \)

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Lagrangian Theory

\[ dw = \alpha_w (w, z) dt + \sqrt{C_{\theta \theta}} dW_z \]

http://www.cmar.csiro.au/airquality/images/LagrangianPlumeModelling.jpg
Theoreticians who advanced application of Lagrangian Modeling to Plant Canopies

John Wilson, Univ Alberta  
Mike Raupach, CSIRO

Theoreticians advancing theory to plant canopies
On the basis of Lagrangian theory Raupach (1987) explains counter-gradient transfer. (because) scalar from nearby elementary sources is dispersing in a near-field regime...its contribution to the overall gradient is much greater than its contribution to the overall flux density.

Just below a fairly localized and intense source in the canopy, the near-field gradient contribution is large and positive; when this is combined with the upward flux of scalar required by conservation of scalar mass, a countergradient flux is obtained.
We study the probability of the trajectory of an ensemble of fluid parcels that are transported and dispersed in the atmosphere. It is a function of the source at a given location and time and the conditional probability of receiving a parcel at a time \( t \) and position \( z \), given it was released at \( z_0 \) and time \( t_0 \). The concentration is the superposition of these trajectories.

\[
C(r, t) = \frac{n(r, t)}{V}
\]

\[
c(z, t) = \int_0^1 \int_0^1 P(z, t \mid z_0, t_0) S(z_0, t_0) dz_0 dt_0
\]

Concentration is function of conditional probability and Source Strength

e.g. the probability of a source released at \( z_0 \) and \( t_0 \) will travel to \( z \) at \( t \)
Lagrangian theory also accounts for non local transport and counter gradient transport. It is the method I use in Can Veg.

For a recent history
This is how a parcel moves an increment \( dx \) given its velocity over a time step \( dt \)
If you look at a plume it will have two limits a near and far field condition.

What is near and far field diffusion? Why do we care? How do we estimate its limits? We will discuss this next.
Gl Taylor, a professor at Cambridge, was one of the leading fluid mechanics of the 20th century. He laid out some of the key theories on plumes

**Statistical Theory of Turbulence**

G. I. Taylor

Let's first look at basics from GI Taylor. His simple derivation is a good way to learn to think about solving problems. $Z$ is height, $W$ is vertical velocity. This is valid for homogeneous turbulence, where there are no gradients in turbulence statistics.
We can derive the variance of the spread of the plume along $z$ as a function of the variance in $w$ and the integral of the Lagrangian correlation coefficient, a measure of persistence of motion. $W$ is Lagrangian velocity, $Z$ is Lagrangian height.
Integrate the correlation coefficient gives the Lagrangian time scale. For short time scales, \( r = 1 \) and \( T_L \) is \( t \), time.
Near field diffusion, cont

\[ \frac{d\sigma_z^2}{dt} = 2\sigma_w^2 t \]

\[ \int d\sigma_z^2 = \int 2\sigma_w^2 t \, dt \]

\[ \sigma_L^2 = \sigma_w^2 t^2 \]

\[ \sigma_L \sim \sigma_w t, t \ll T_L \]

Diffusion scale is proportional to travel time

This allows us to define the spread of the plume in the near field when \( t \ll T_L \). It is proportional to the standard deviation in \( W \) times time, \( t \).
Far field is when time goes to infinity
Far field diffusion, cont

\[
\int d\sigma_z^2 = \int 2\sigma_w^2 T_L \, dt
\]

\[
\sigma_z^2 = 2\sigma_w^2 T_L t
\]

\[
\sigma_z = \sigma_w \sqrt{2T_L t}
\]

Diffusion is proportional to the square root of travel time

Here spread is proportional to the square root of time at the far field limit
Plume is tight near source, concentrations elevated
Langevin equation is the foundation for studying parcel movement. It has a persistent (-a u) and random term. The next task is to define the a and b coefficients.
\[ \xi \quad \text{Gaussian White Noise} \]
- Uniform spectral density
- Gaussian pdf
- Mean of zero
- Variance of \( \text{dt} \)
- Uniform spectral density
- Discontinuous

\[ W(T) = \int_0^T \xi(t) \, dt \quad \text{The integral is continuous and defined} \]

\[ \frac{dW(t)}{dt} \quad \text{The time derivative of the random component is not defined} \]

\[ dW(t) = \xi(t) \, dt \quad \text{The increment is defined as an incremental} \quad \text{Weiner process} \]
Rules of Weiner Process

\[ \langle dW(t) \rangle = 0 \]

\[ \langle |dW(t)|^2 \rangle = dt \]

\[ \langle (W(t) - w_0) \rangle = t - t_0 \]
Defining Coefficients of Langevin Equ

\[ \frac{du}{dt} = -a_i u + b \xi(t) \]

\[ \alpha = \frac{1}{T_L} \quad b = \sigma_w \sqrt{2a_i} = \sigma_w \sqrt{2/T_L} \]

\[ dw = -\frac{w}{T_L} dt + \sqrt{\frac{2\sigma_w^2}{T_L}} \xi(t) dt \]
If turbulence is heterogeneous an extra term arises and is necessary. This is what happens in canopies where there are strong gradients in variance, time scales, shear, etc. If you apply Langevin equation in a canopy it will numerically violate laws of physics. A well mixed field should remain well mixed. But with the Langevin equation there will be a build up of particles near the floor as the drop down more then up due to asymmetric turbulence. So new terms need to be found
Thomson’s Theory

- the well-mixed criterion must hold: if particles are initially well-mixed they will remain so in turbulent flow
- Eulerian models derived from the Lagrangian model must be compatible with Eulerian equations that are derived from first principles.
- the model will reduce to the diffusion equation when the Lagrangian time scale approaches zero
- the forward and reverse formulations of dispersion are consistent
- the small time behavior of the velocity distribution of particles must be correct.


Thomson developed one of the leading theories for applying the Lagrangian model in heterogeneous turbulence using the Fokker Planck Equation
This is a partial differential equation for a conditional probability. This type of stochastic partial differential equation is used in the ED and ED2 model for forest dynamics.
Simple version of Fokker Planck limits leads to Gaussian plume
Translation

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spreading

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We can compute $D_{ij}$ for a known source of unit strength by calculating the $C$ fields with a Lagrangian model by releasing parcels from different heights. Then apply it to compute $C$ fields by iterating with $S(c)$ for real conditions.
Here is example of Dispersion matric
Dij is noisy with fewer parcels
Analytical Near/Far Field Theory of Raupach

\[ C(z) = C_n(z) + C_f(z) \]

\[ \frac{\partial}{\partial z} (K(z)) \frac{\partial C_f(z)}{\partial z} = -S(z) \]

\[ F_s(z) = -K(z) \frac{\partial C_f(z)}{\partial z} = F_i + F_s \]

\[ F_s(z) = \int_0^z S(z')dz' \]

\[ K(z) = \sigma_v^2(z) T_z(z) \]

Analytical models of Raupach

Simple assumptions of canopy turbulence. One can use empirical fits to data or use a higher order closure model to produce estimates of turbulence statistics given the canopy leaf area index, height, leaf size, for a hybrid Eulerian-Lagrangian model.
$$C_f(z_r) = C(z_r) - C_n(z_r)$$

$$C_f(z) - C_f(z_r) = \int_{z_r}^{z} \frac{F(z)}{K_f} dz$$

$$C_f(z) - C_f(z_r) = \int_{z_r}^{z} \frac{1}{\sigma_n(z') T_L(z')} \left[ \int_{z_r}^{z} S(z'') dz'' + F_s \right] dz'$$

$$C_s(z) = \int_{-\infty}^{z} S(z') \left( k_n \frac{z-z'}{\sigma_n(z') T_L(z')} + k_s \frac{z+z'}{\sigma_n(z') T_L(z')} \right) dz'$$

$$k_n(x) \approx -0.3984 \ln(1 - \exp(-|x|)) - 0.1562 \exp(-|x|)$$
Improvements Warland and Thurtell

\[ \frac{\partial c}{\partial z} = \int_0^\infty \frac{-q}{\sqrt{2\pi \sigma_z^2}} \frac{(z-z_i)}{\sigma_z^2} \exp\left(-\frac{(z-z_i)^2}{2\sigma_z^2}\right) \]

\[ \frac{\partial c}{\partial z} \bigg|_{ff} = \frac{-q}{2\sigma_v L_z} \quad \frac{\partial c}{\partial z} \bigg|_{nf} = \frac{-q}{\sqrt{\pi \sigma_v} |\tau - \tau_i|} \]

\[ \frac{\partial c}{\partial z} - Dq \]

\[ D_{ff} = \begin{cases} 
-1 \quad &; z > z_i \\
\frac{1}{2\sigma_v L_z} \quad &; z < z_i
\end{cases} \]

\[ D_{nf} = \begin{cases} 
\frac{1}{\sqrt{2\pi \sigma_v} (z-z_i)} \quad &; z > z_i \\
\frac{1}{\sqrt{2\pi \sigma_v} (z-z_i)} \quad &; z < z_i
\end{cases} \]
Vertical Gradients in CO₂

Graph showing the vertical gradients in CO₂ with height and CO₂ concentration.