Linear Programming (LP)

Set up

Facts

Dual

1. **Objective**: Variables

\[ x_1, \ldots, x_n \in \mathbb{R}^+ \]

acres of land used

extent of a type harvested

used in treatment \( i \)

Real \( \# 's \) often \( x > 0 \)

2. **Objective**

\[ z = c_1 x_1 + \ldots + c_n x_n \quad \text{for LP} \]

thing you want to maximize

Can have only one objective

yes: max. \( pV \) of harvest \( + \) \( R \) for \( x \) \( \times \) \( \text{acres} \)

in late season

no: max. both \( S \) and \( D \)

no: greatest \( S \) and \( D \) for greatest \( z \)

\[ z = c_1 x_1 + c_2 x_2 \quad \text{(two dimensions)} \]

\[ x_2 = (z - c_1 x_1) c_2 \]

\( \leftarrow \) higher line is

\( \leftarrow \) iso-\( z \) lines both direction of fastest increase

2/c_1
Constraints

\[ a_1 x_1 + \ldots + a_n x_n = b_1 \]

\[ \vdots \]

\[ a_m x_1 + \ldots + a_n x_n = b_m \]

\[ m \text{ constraints} \]
\[ n \text{ variables} \]

\[ \mathbf{x} = (x_1, \ldots, x_n) \text{ that satisfies all } \]
\[ m \text{ constraints is called feasible.} \]
\[ (\mathbf{x} \geq 0, \text{ must also satisfy those constraints}) \]

All the \( \mathbf{x} \) that are feasible
on the feasible set \( \{ \mathbf{x} \mid \mathbf{x} \text{ is feasible} \} \)

What of

\[ a_1 x_1 + \ldots + a_n x_n \leq b_1 \]?

Make a big problem by adding
\[ an \quad x_{n+1} > 0. \text{ Note that} \]
\[ c_{n+1} = 0 \text{ so it doesn't affect} \]
\[ \text{objective} \]
\[ a_{11} x_1 + \ldots + a_{1n} x_n + x_{n+1} = b_1 \]

and so on
\[ x_{n+1} \text{ is called a slack variable} \]
and often called \( S \),
\[ a_{1n} x_1 + \ldots + a_{1n} x_n + S = b \]
Compact Matrix notation

\[
\max z = c'x \\
\text{s.t. } Ax \leq b \\
\text{and (maybe) } x \geq 0.
\]

This is standard \textit{max} problem.

Example: Keith's poet's words.

\[x_1 \text{ is } \text{manured acres ha} \]
\[x_1 \text{ red pine } x_2 \text{ hardwood} \]

\[\text{cons 1: } x_1 < 40 \text{ ha} \]
\[\text{cons 2: } x_2 \leq 50 \text{ ha} \]

Each ha of red pine takes 2 hrs to
\[\text{manure, each ha of hardwood} \]
\[\text{takes 3 hours. There are} \]
\[180 \text{ hours available.} \]

\[\text{cons 3: } 2x_1 + 3x_2 \leq 180 \]
\[\text{non neg. } x_1, x_2 \geq 0 \]

Objective: \$90 \text{ for each acre pine} 
\[120 \text{ for each acre hardwood} \]
\[z = 90x_1 + 120x_2 \]
\[c_1 = 90 \quad c_2 = 120 \]
Let's rewrite Keith's problem:

\[
\begin{align*}
\max & \quad 90x_1 + 120x_2 \\
& \quad 1x_1 + 0x_2 \leq 40 \\
& \quad 0x_1 + 1x_2 \leq 50 \\
& \quad 2x_1 + 3x_2 \leq 180 \\
\end{align*}
\]

OR

\[
\begin{align*}
\begin{bmatrix} 90 & 120 \\ 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 40 \\ 50 \\ 180 \end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\max_{x} & \quad c^Tx = 2 \\
\text{Ax} & \leq b
\end{align*}
\]

Now let's draw a picture of the feasible set: \((x_1, x_2 \geq 0)\)
What does a maximum look like?

Every maximum includes corners. Only need to examine corners to find answer.

Corners in 2 dimensions are intersection of 2 lines.

In general, n unknowns means n possible (< m) constraints determine a corner.
At this corner only \( x_2 = 50 \) and \( 2x_1 + 3x_2 = 180 \) determine the corner. \( x_1 < 40 \), so that constraint isn't needed. It is said to be slack.

\( x_2 = 50 \) and \( 2x_1 + 3x_2 = 180 \) are binding.

\( n \) dimensions \( \rightarrow n \) binding constraints
\( m-n \) slack constraints

Now for some algebra... 

\( x_2 = 50 \)

\[ 2x_1 + 3x_2 = 2x_1 + 3 \cdot 50 = 180 \]

\[ x_1 = \frac{30}{2} = 15 \]

\[ z = 90 \cdot 15 + 120 \cdot 50 \]

\[ 1350 + 6000 = 7350 \]
Let's look at the next corner:

\[ x_1 = 40 \]
\[ 2(40) + 3x_2 = 180 \]
\[ x_2 = 10 \]
\[ 90 \cdot x_1 + 120 \cdot x_2 = 2 \]
\[ 3600 + 4000 = 7600 \]

This is the best corner.
(You could check the other two to be complete.)

In Algebra - same thing exactly:

\[
\begin{pmatrix}
1 & 0 \\
2 & 3
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} =
\begin{pmatrix}
40 \\
180
\end{pmatrix}
\]

\[
\begin{pmatrix}
A_{13}
\end{pmatrix} \begin{pmatrix}
\hat{x}
\end{pmatrix} = b
\]
\[
\begin{pmatrix}
\hat{x}
\end{pmatrix} = A_{13}^{-1} b = \begin{pmatrix}
40 \\
100/3
\end{pmatrix}
\]
Dual Variables or Shadow Prices

How much would it be worth to relax a constraint by 1 unit?

E.g. suppose \( x < 40 + 1 \)

Answer: the dual or shadow variable \( \lambda \).

Method: let's find two numbers (1 for each binding constraint) \( \lambda_1 \) and \( \lambda_3 > 0 \). At these prices we want our optimal policy to just break even and all other policies to lose money.

\( \lambda_3 \) because it is the 3\textsuperscript{rd} constraint.

We have \( x_1 = 40 \) and \( x_2 \).

A ha of pine (unit of \( x_1 \)) requires
1 ha of pine.

Recall \( 1x_1 + 0x_2 = 40 \)

It also requires 2 hrs of work
It pays \$50

\( \lambda_1 \cdot 1 + \lambda_3 \cdot 2 = 50 \)

for break even.
Now let's look at the second case.

We call growing pine and growing hardwoods activities.

It uses up 100 pine and 2 hrs labor so

\[ 2 \cdot 0 + 2 \cdot 3 = 120 \]

\[
\begin{pmatrix}
90 \\
10 \\
23
\end{pmatrix}
\]

We are going to look at the columns.

\[
(2, 2) \begin{pmatrix}
1 & 0 \\
2 & 3
\end{pmatrix} = (90, 120)
\]

In matrix form

Let's solve it

\[ 3 \lambda_2 = 120 \quad \lambda_2 = 40 / 3 \]

\[ \lambda_1 + 2 \lambda_2 = 90 \]

\[ \lambda_1 + 80 = 90 \]

\[ \lambda_1 = 10 \]

\[ \tilde{\lambda} A_{13} = c \]

Symbolically \( \tilde{\lambda} = c A_{13}^{-1} \).
Does this make sense?

Let's look at

\[
\begin{pmatrix}
1 & 0 \\
2 & 3
\end{pmatrix}
\begin{pmatrix}
\chi_1 \\
\chi_2
\end{pmatrix} =
\begin{pmatrix}
41 \\
120
\end{pmatrix}
\]

\[2 \cdot \chi_1 + 3 \cdot \chi_2 = 120\]

\[\frac{\chi_2}{3} = \frac{98}{3}\]

\[\chi_2 = \frac{98}{3}\]

\[\chi_1 = 90 - 41 + \frac{98}{3} \cdot 120\]

\[3600 + 90 + \frac{100}{3} \cdot 120 - \frac{2}{3} \cdot 120 = 7600 + 70\]

\[\lambda_1 = 10 + \text{we do gain 10.}\]

Add \(\chi_1\) requires 2 more units labor

Lose \(\frac{2}{3}\) unit of \(\chi_2\)

Gain $50, Lose \frac{2}{3} \cdot 120 = 80 \text{ net 10}$
Here we have used the slack variables $S > 0$ to convert the inequality to equality.

Complementary slackness

$$S_i \cdot \lambda_i = 0$$

Either constraint is tight

$$S_i = 0$$

or the constraint has a slack price of zero

$$\lambda_i = 0$$
Dual + Primal same value

Recall \( \bar{x} = A_{13}^{-1} b \) was the optimal choice of \( \bar{x} \)

\[ c' \bar{x} = c A_{13}^{-1} b \] is the value

But \( \bar{\lambda} = c A_{13}^{-1} \)

So \( c \bar{x} = \bar{\lambda} b \)

\[
\begin{pmatrix} 10 & 40 \\ 40 & 180 \end{pmatrix} = \frac{8}{7600}
\]

which works

Shadow price

Value if \( b \) increases a little

Say \( b \rightarrow (b_1 + 1) \)

\( \bar{\lambda} \)

\( b \rightarrow (b_1) \)

\( \bar{\lambda} \)
So what's the use?

\[ I_2 = \$40/hr \]

Maybe the poet should do more forestry as his next several hours will earn $40 each and poets aren't rich.
Two Stand - Base Model

\( x_{11}, x_{12}, x_{13} \) cats in 3 periods

in acres from first stand

\( x_{21}, x_{22}, x_{23} \) from second stand

\[ \begin{array}{ccccccc}
B & A & G & p & S & N \\
8 & 2 & 16 & 23 & 33 & 24 & 32 & 45 \\
36 & 1 & 1 & 0 & 0 & 0 & 120 \\
45 & 0 & 0 & 0 & 1 & 1 & 1 & 120 \\
-12 & 1 & 1 & 0 & 0 & 1 & 100 \\
-13 & 0 & 1 & 1 & 0 & 1 & 100 \\
\end{array} \]

\( C \) is tons of polynomial

\[ \begin{array}{c}
\text{Solve} \\
\text{Area converted in ha} \\
\text{Comp } 1 \ 2 \ 3 \\
1 & 100 & 20 & 0 & = 120 \\
2 & 0 & 80 & 100 & = 180 \\
\end{array} \]

Dual

\[ \begin{align*}
\lambda_1 + \lambda_3 &= 16 \\
\lambda_2 + \lambda_4 &= 23 \\
\lambda_2 + \lambda_4 &= 32 \\
\lambda_2 &= 45 \\
\lambda_4 &= \frac{1}{2} (13 - \lambda_1) \\
\lambda_1 &= 36 \\
\lambda_3 &= -20
\end{align*} \]
The activity not taken.

Let's try $x_3$

price it out

$33 - \lambda_1 = 33 - 36 = -3$

Let's try $x_{21}$

$24 - \lambda_2 - \lambda_3 = 24 - 45 + 20 = -1$

The rule

$2A \succeq c$

columns of matrix called activities

$(2 \begin{bmatrix} A_{11} & \cdots & A_{1n} \end{bmatrix} - c \geq 0$

in our example

columns 1, 2, 5, 6 =

columns 3, 4, 7

Complementary Slackness

$(2A - c)x = 0$

in our example $x = x_{13} = 0$

and $2A - c \geq 0$ for them