Linear Programming. (LP)
Set up
Facts
Dual

1. ~~Objective~~ Variables

$x_1 \ldots x_n \in R^+$

acres of land ~~used~~ extant harvested
of a type
used in treatment i

Real #'s often $x \geq 0$

2. Objective

$z = c_1 x_1 + \ldots + c_n x_n$ for LP

Thing you want to maximize

Can have only one objective

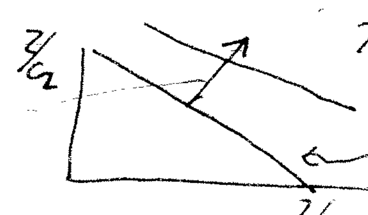
yes: max PV of harvest + 10,000 * # acres in late seral stage

no: max both \$ and diversity

no: greatest good for greatest #

$z = c_1 x_1 + c_2 x_2$ (two dimension)

$x_2 = (z - c_1 x_1) c_2$



z/c_2

z/c_1

direction of fastest increase

← higher line is better

← iso-z lines

Constraints

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

m constraints
n variables

An $\vec{x} = (x_1 \dots x_n)$ that satisfies all m constraints is called feasible.
(if $x > 0$, must also satisfy those constraints)

All the ~~$(x_1 \dots x_n)$~~ $\vec{x}$ that are feasible are the feasible set. $\{\vec{x} \mid \vec{x}$ is feasible

What of

$$a_{11}x_1 + \dots + a_{1n}x_n \le b_1 \ ?$$

Make a bigger problem by adding an $x_{n+1} > 0$. Note that $c_{n+1} = 0$ so it doesn't affect objective

$$a_{11}x_1 + \dots a_{1n}x_n + x_{n+1} = b_1$$

and so on

x_{n+1} is called a slack variable and often called s_1

$$a_{11}x_1 + \dots a_{1n}x_n + s_1 = b_1$$

Compact Matrix notation

$$\max z = c'\vec{x}$$
$$\text{s.t. } A\vec{x} \leq b$$
$$\text{and (maybe) } \vec{x} > 0.$$

This is standard max problem.

Example: Keith's poets woods.

x_i is managed ~~acres~~ ha

x_1 red pine x_2 hardwoods

cons 1 $x_1 < 40$ ha   what there is

cons 2 $x_2 \leq 50$ ha

Each ha of red pine takes 2hs to manage. each ha of hardwood takes 3 hours. There are 180 hours available.

cons 3 $2x_1 + 3x_2 \leq 180$

non neg $x_1, x_2 \geq 0$

Objective: \$90 for each acre pine
120 for each acre hardwood

$$z = 90x_1 + 120x_2$$
$$c_1 = 90 \quad c_2 = 120$$

Lets rewrite Keiths problem

$$\max \; 90x_1 + 120x_2$$

$$1x_1 + 0x_2 \le 40$$

$$0x_1 + 1x_2 \le 50$$

$$2x_1 + 3x_2 \le 180$$

or

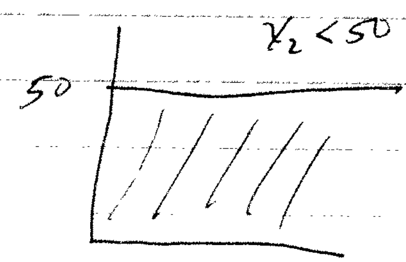
$$\begin{pmatrix} 90 & 120 \end{pmatrix} = z$$

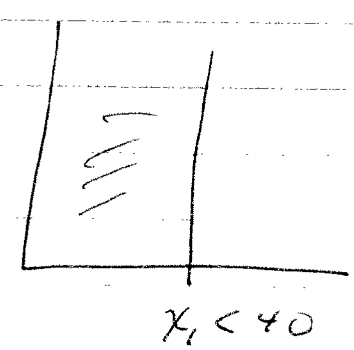
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le \begin{pmatrix} 40 \\ 50 \\ 180 \end{pmatrix}$$

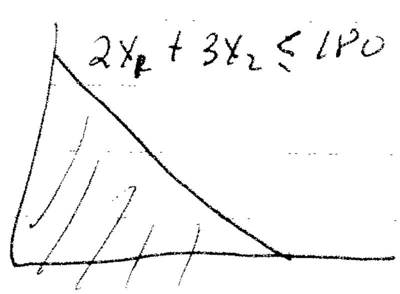
$$\max_x \; c'x = z$$

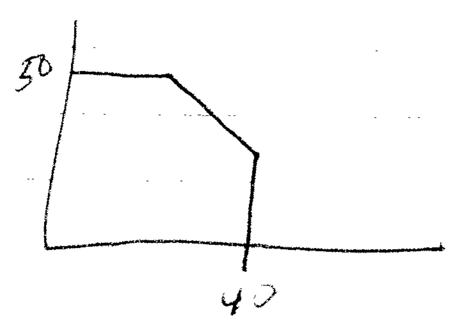
$$Ax \le b$$

Now lets draw a picture of the feasible set. $(x_1, x_2 \ge 0)$

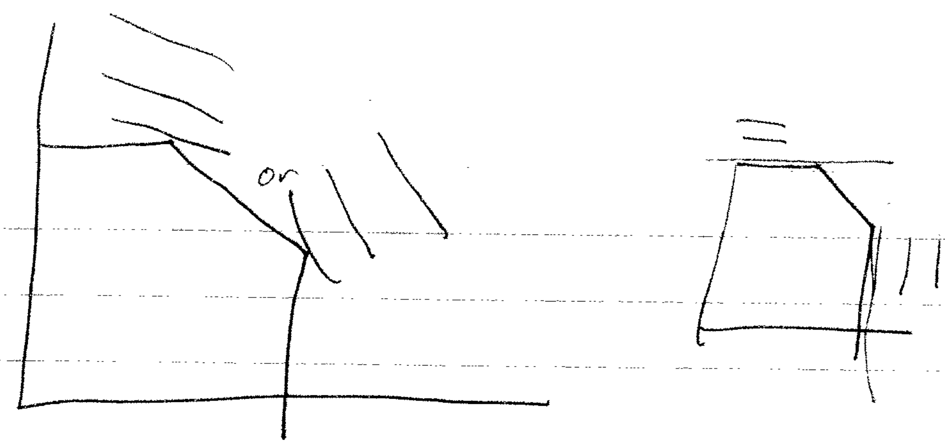


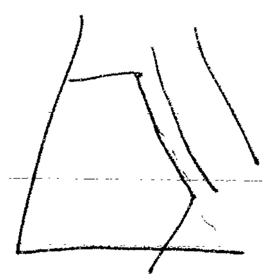






What does a maximum look like?

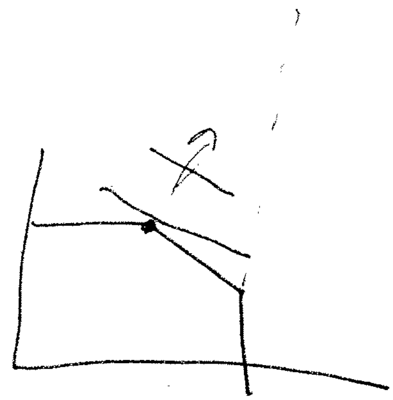




Every maximum includes <u>corners</u>.
Only need to examine corners to find answer.

Corners in 2 dimensions are intersection of 2 lines

In general n unknowns means n possible ($< m$) constraints determine a corner



At this corner only $x_2 = 50$ and $2x_1 + 3x_2 = 180$ determine the corner. $x_1 < 40$ so that constraint isn't needed. It is said to be <u>slack</u>.
$x_2 = 50$ and $2x_1 + 3x_2 = 180$ are binding.

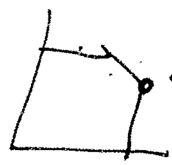
n dimensions → n binding constraints
m−n slack constraints

Now for some algebra ...

$$x_2 = 50$$
$$2x_1 + 3x_2 = 2x_1 + 3 \cdot 50 = 180$$
$$x_1 = 30/2 = 15$$

$$z = 90 \cdot 15 + 120 \cdot 50$$
$$1350 + 6000 = 7350$$

Lets look at the next corner



← this one

$$x_1 = 40$$
$$2(40) + 3x_2 = 180$$
$$x_2 = 100/3$$
$$90 \cdot x_1 + 120 \cdot x_2 = z$$
$$3600 + 4000 = 7600$$

This is the best corner.
(You could check the other two to be complete.)

In Algebra – same thing exactly

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 180 \end{pmatrix}$$

$$(A_{13})\,\vec{x} = b$$

$$\vec{x} = A_{13}^{-1}\, b = \begin{pmatrix} 40 \\ 100/3 \end{pmatrix}$$

Dual Variables or Shadow prices

How much would it be worth to relax a constraint by 1 unit.

e.g. suppose $x < 40+1$

Answer: the dual or shadow variable λ_1

Method: Lets find two numbers (1 fore each binding constraint) λ_1 and $\lambda_3 > 0$. At these prices we want our optimal policy to just break even and all other policies to ~~loose~~ lose money.

(λ_3 because it is the 3rd constraint

~~We use $x_1 = 40$ and $x_2 = 100/3$~~

A ha of pine (unit of x_1) requires 1 ha of pine.

Recall $1x_1 + 0x_2 = 40$

It also requires 2 hrs of work

It pays \$90

$$\lambda_1 \cdot 1 + \lambda_3 \cdot 2 = \$90$$

for breakeven

Now lets look at the second ~~case~~ ~~piece~~ activity (we call growing pine and growing hardwoods activities)

It uses up no pine and 2 hrs labor so

$$\lambda_1 \cdot 0 + \lambda_3 \cdot 3 = 120$$

$$\begin{matrix} 90 & 120 \\ 1 & 0 \\ 2 & 3 \end{matrix}$$

We are going to look at the colums.

$$\underset{\lambda}{(\lambda_1, \lambda_2)} \underset{A_{13}}{\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}} = \underset{c}{(90, 120)}$$

in matrix form

lets solve it

$$3\lambda_3 = 120 \qquad \lambda_2 = 40$$

$$\lambda_1 + 2\lambda_3 = 90$$

$$\lambda_1 + 80 = 90$$

$$\lambda_1 = 10$$

$$\vec{\lambda} A_{13} = c$$

Symbolically $\vec{\lambda} = c A_{13}^{-1}$

Does this make sense?
lets look at

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{matrix} 41 \\ 180 \end{matrix}$$

$$2 \cdot 41 + 3 \cdot x_2 = 180$$

$$x_2 = 98/3$$

$$z = 90 \cdot 41 + \frac{98}{3} \cdot 120$$

$$=$$

$$3600 + 90 + \frac{100}{3} 120 - \frac{2}{3} 120 = 7600 + 10$$

$\lambda_1 = 10$ + we do gain 10.

Add x_1 requirs 2 more units labor
lose $2/3$ unit of x_2

gain \$90 lose $2/3 \cdot 120 = 80$ net 10

Pivot Again

		x_1	x_2	s_1	s_2	s_3		
		90	120	0	0	0		z
λ_1	10	1	0	1	0	0	=	40
λ_2	0	0	1	0	1	0		50
λ_3	40	2	3	0	0	1		180

Here we have used the slack variables $s > 0$ to convert the inequality to equality

Complementary Slackness

$$s_i \lambda_i = 0$$

either constraint is tight

$s_i = 0$

or the constraint has a shadow price of zero

$\lambda_i = 0$

Dual + Primal Same Value

Recall $\vec{x} = A_{13}^{-1} b$ was the optimal choice of $\vec{x}$

$c^{tr} \vec{x} = c A_{13}^{-1} b$ is the value

But $\lambda = c A_{13}^{-1}$

So $c \vec{x} = \vec{\lambda} b$

$(10, 40) \begin{pmatrix} 40 \\ 180 \end{pmatrix} = \$7600$

which works

Shadow price

Value if b increases a little

Say $b \Rightarrow \begin{pmatrix} b_1 + 1 \\ b_2 \end{pmatrix}$

is λ_1

$b \to \begin{pmatrix} b_1 \\ b_2 + 1 \end{pmatrix}$

is λ_2

So what's the use?

$\lambda_2 = \$40/hr$ Maybe the poet should do more forestry as his next several hours will earn \$40 each and poets aren't rich.

Two Stand - 3age Model

x_{11}, x_{12}, x_{13} cuts in 3 periods in acres from first stand

x_{21}, x_{22}, x_{23} from second stand

B-G p59

λ		x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	
	z	16	23	33	24	32	45	
36	Comp1	1	1	1	0	0	0	= 120
45	comp2	0	0	0	1	1	1	= 180
~~-13~~ -20	AGC1	1	0	0	1	0	0	100
-13	AGC2	0	1	0	0	1	0	100
Redundant	~~AGC3~~	~~0~~	~~0~~	~~1~~	~~0~~	~~0~~	~~1~~	

z is tons of pulpwood

Soln Area converted in ha

Comp	1	2	3	
1	100	20	0	= 120
2	0	80	100	= 180

Dual

$$\lambda_1 + \lambda_3 = 16$$

$$\lambda_1 + \lambda_4 = 23$$

$$\lambda_2 + \lambda_4 = 32$$

$$\lambda_2 = 45$$

$\therefore$ $\lambda_4 = -13$ $\quad \lambda_1 = 36$ $\quad \lambda_3 = -20$

The activity not taken: 15

Lets try x_{13}
price it out

$\$33 - \lambda_1 = 33 - 36 = -3$

Lets try x_{21}

$\$24 - \lambda_2 - \lambda_3 = 24 - 45 + 20 = -1$

The rule

$\lambda A \geq c$

$$\Big(\lambda \underbrace{\left[A_{\cdot 1}, \ldots A_{\cdot n} \right]}_{\text{columns of matrix called activities}} - c \Big) \geq 0$$

in our example

Columns $1, 2, 5 6$ $=$

Columns $3, 4$ $>$

~~Complementar~~ Complementary Slackness

$(\lambda A - c) x = 0$

In our example $x_{24} = x_{13} = 0$

and $\lambda A - c > 0$ for them